Mid-Term Exam

Geometry/Topology I

Solution Hints

Do any **four** of the following problems. 25 points each.

1. Let (X, d) be a connected metric space containing more than one point. Prove that X is uncountable.

Let x and y be two distinct points of X, and set $D = d(x, y) > 0$. Let $f : X \to \mathbb{R}$ be the continuous function defined by $f(z) = d(x, z)$. Since X is connected, $f(X)$ is connected, too. Hence $[0, D] \subset f(X)$. Thus $f(X)$ is uncountable. Hence X must be uncountable, too.

2. Let $f : X \to Y$ be an injective continuous map between topological spaces. If X is compact and Y is Hausdorff, prove that X is homeomorphic to $f(X) \subset Y$.

The key point is that f is a closed map. To see this, suppose that $A \subset X$ is closed. Since X is compact, it follows that A is also compact. Hence $f(A)$ is compact. Since Y is Hausdorff, this implies that $f(A) \subset Y$ is closed. Moreover, $f(A) \subset f(X)$ is closed in the subspace topology. The induced map $X \to f(X)$ is therefore a closed, continuous bijection, and so is a homeomorphism.

3. Let $\mathbb{R}^n$ be given its usual Euclidean metric d and its usual topology. If $A \subset \mathbb{R}^n$ is a non-empty closed set, and if $x \in \mathbb{R}^n$ is any given point, prove that there is a point $y \in A$ of minimum distance from x. In other words, show that

$$\exists y \in A \text{ such that } d(x, z) \geq d(x, y) \quad \forall z \in A.$$

Extra credit for pointing out the omitted hypothesis!

Given some $w \in A$, let $R = d(x, w)$, and observe that $C = A \cap \overline{B_R(x)} \subset \mathbb{R}^n$ is closed and bounded. Thus C is compact by the Heine-Borel theorem, and it is also non-empty, since $w \in C$. The continuous function $f : C \to \mathbb{R}$ defined by $f(z) = d(x, z)$ therefore achieves its minimum value at some $y \in C \subset A$, and this is the desired closest point.

4. Let X be a compact topological space, and let $A_1 \supset A_2 \supset \cdots \supset A_j \supset \cdots$ be a nested sequence of closed, non-empty subsets of X. Prove that $\cap_{j=1}^{\infty} A_j \neq \emptyset$.

Suppose not. Then the open sets $U_j = X - A_j$ cover X. Since X is compact, only finitely many are needed, so $X = U_1 \cup \cdots \cup U_n$ for some n. But since $U_j \subset U_{j+1}$, this implies that $X = U_n$. However, $X - U_n = A_n \neq \emptyset$, so this is a contradiction.

5. Let $U \subset \mathbb{R}$ be an open set. Prove that U is a countable union of **disjoint** open intervals. (**Hint:** put off the issue of countability until the very end.)

Let $I_\alpha \subset U$, $\alpha \in J$, be the connected components of U. Then each I_α is open in U, because U is locally (path) connected. Since U is open in $\mathbb{R}$, this means that each I_α is a connected open subset of $\mathbb{R}$, and so is an open interval. Thus, $U = \bigcup_{\alpha \in J} I_\alpha$ is a disjoint union of open intervals.
To show that the index set J must be countable, now observe that each I_α must meet the rationals $\mathbb{Q}$, since $\mathbb{Q} \subset \mathbb{R}$ is dense. Choose one rational number q_α from each I_α. Since the intervals I_α are disjoint, the numbers q_α are all different, and the function $J \to \mathbb{Q}$ given by $\alpha \mapsto q_\alpha$ is therefore injective. Thus J is bijectively equivalent to a subset of a countable set, and is therefore countable.

6. Let X and Y be topological spaces, and assume that Y is Hausdorff. Let $f, g : X \to Y$ be two continuous functions. Prove that $\{x \in X \mid f(x) = g(x)\}$ is a closed subset of X.

Set $A = \{x \in X \mid f(x) = g(x)\} \subset X$.

Method #1 (Elementary). Suppose $x \notin A$. Then $f(x) \neq g(x)$. Since Y is Hausdorff, $\exists U, V \subset Y$ disjoint open sets with $f(x) \in U$, $g(x) \in V$. Set $W = f^{-1}(U) \cap g^{-1}(V)$. Then W is open, with $x \in W \subset (X - A)$. Hence $X - A$ is open, and A is therefore closed.

Method #2 (Sophisticated). Let $\Delta \subset Y \times Y$ be the diagonal $\{(y, y) \mid y \in Y\}$. Since Y is Hausdorff, Δ is closed. However, $f \times g : X \to Y \times Y$ is continuous. Thus $A = (f \times g)^{-1}(\Delta)$ is also closed, exactly as claimed.

7. Let X be a compact Hausdorff space. Prove that X is normal ($\mathbf{T}_4$).

See Munkres, page 202.

8. Let X and Y be regular ($\mathbf{T}_3$) topological spaces. Prove that $X \times Y$ is regular, too. (Hint: recall that the $\mathbf{T}_3$ axiom can be reformulated in terms of open sets.)

If $W \subset X \times Y$ is open, and if $(x, y) \in W$ is any point, we need to show that (x, y) has an open neighborhood U with $\overline{U} \subset W$. However, any such W is a union of basis open sets. So there exist open sets $W_1 \subset X$ and $W_2 \subset Y$ with $(x, y) \in W_1 \times W_2 \subset W$. But since X is regular, we can find open neighborhoods V_1 of x and V_2 of y such that $\overline{V}_j \subset W_j$, $j = 1, 2$. Moreover, $\overline{V}_1 \times \overline{V}_2$ is closed in $X \times Y$, since its complement $[(X - \overline{V}_1) \times Y] \cup [X \times (Y - \overline{V}_2)]$ is open. Hence

$$(x, y) \in V_1 \times V_2 \subset \overline{V_1 \times V_2} \subset \overline{V}_1 \times \overline{V}_2 \subset W_1 \times W_2 \subset W$$

and so $U = V_1 \times V_2$ fulfills our requirement, and $X \times Y$ is regular, as claimed.

Remark. In fact, $\overline{V_1 \times V_2} = \overline{V}_1 \times \overline{V}_2$, but you do not need this here.

9. Let (X, d) be a metric space, and let A and B be disjoint closed sets. Using d, construct an explicit continuous function $f : X \to [0, 1]$ such that $A = f^{-1}(0)$ and $B = f^{-1}(1)$. Then relate and compare this to the Urysohn lemma.

One such function is

$$f(x) = \frac{d_A(x)}{d_A(x) + d_B(X)}$$

where $d_A(x) = \inf\{d(x, y) \mid y \in A\}$ and $d_B(x) = \inf\{d(x, y) \mid y \in B\}$. In particular, X is $\mathbf{T}_4$, since $U = f^{-1}((-\infty, \frac{1}{4}))$ and $V = f^{-1}((\frac{3}{4}, \infty))$ are disjoint open sets which contain A and B, respectively. However, the constructed f is better than the one promised by the Urysohn Lemma for an arbitrary $\mathbf{T}_4$ space, since we actually have $A = f^{-1}(0)$ and $B = f^{-1}(1)$, rather than just $A \subset f^{-1}(0)$ and $B \subset f^{-1}(1)$.