MAT 132 Exam # 2 Answer Key

1. Find the average value of the function

\[ f(x) = \sin^2 x \]

on the interval \( 0 \leq x \leq \pi \).

\[
\frac{1}{\pi - 0} \int_{0}^{\pi} \sin^2 x \, dx = \frac{1}{\pi} \int_{0}^{\pi} \frac{1 - \cos 2x}{2} \, dx
\]

\[
= \frac{1}{2\pi} \int_{0}^{\pi} (1 - \cos 2x) \, dx
\]

\[
= \frac{1}{2\pi} \left[ x - \frac{\sin 2x}{2} \right]_{0}^{\pi}
\]

\[
= \frac{1}{2\pi} \left[ (\pi - 0) - (0 - 0) \right]
\]

\[
= \frac{1}{2}
\]

Average value = \( \frac{1}{2} \)

Answer
2. (a) Suppose that, for some constant $A$, the probability distribution for a certain continuous random variable $x$ is given by

$$f(x) = \frac{A}{1 + x^2}.$$ 

What is the value of $A$?

A probability distribution $f(x)$ must satisfy

$$\int_{-\infty}^{\infty} f(x) \, dx = 1.$$ 

Since

$$\int_{-\infty}^{\infty} \frac{A}{1 + x^2} \, dx = \lim_{M \to -\infty} \left[ A \tan^{-1} x \right]_M^0 + \lim_{L \to \infty} \left[ A \tan^{-1} x \right]_0^L = A \left[ 0 - (-\frac{\pi}{2}) \right] + A \left[ \frac{\pi}{2} - 0 \right] = A \pi,$$

we must therefore have $A \pi = 1$. Hence $A = 1/\pi$.

$$A = \frac{1/\pi}{\pi} = \frac{1}{\pi}$$

(b) What is the probability that $x$ is between 0 and 1?

$$P(0 \leq x \leq 1) = \int_0^1 f(x) \, dx = \int_0^1 \frac{A}{1 + x^2} \, dx = \int_0^1 \frac{1}{\pi} \, dx = \frac{1}{\pi} \int_0^1 \frac{1}{1 + x^2} \, dx = \frac{1}{\pi} \left[ \tan^{-1} x \right]_0^1 = \frac{1}{\pi} \left[ \frac{\pi}{4} - 0 \right] = \frac{1}{4}$$

$$P(0 \leq x \leq 1) = \frac{1}{4}$$
3. The diagram below depicts the \textit{direction field} corresponding to a certain differential equation

\[
\frac{dy}{dx} = f(x, y).
\]

(a) Sketch the solution of the initial value problem

\[
\frac{dy}{dx} = f(x, y), \quad y(0) = 0,
\]

on the diagram above.

(b) The relevant differential equation is actually one of the equations listed below. Which one? Circle your answer.

(i) \[
\frac{dy}{dx} = \tan^{-1} x
\]

(ii) \[
\frac{dy}{dx} = 1 - 2[1 + e^{2x}]^{-1}
\]

(iii) \[
\frac{dy}{dx} = 1 + y^2
\]

(iv) \[
\frac{dy}{dx} = x^2y^2
\]

(v) \[
\frac{dy}{dx} = 1 - y^2
\]

because picture shows that \(f(x, y)\) only depends on \(y\), and \(= 0\) for at least two values of \(y\).
4. Find the general solution $y(x)$ of the differential equation

$$\frac{dy}{dx} = x^3 y^2$$

Give your final answer in explicit form.

$$\frac{dy}{y^2} = x^3 \, dx$$

$$\int \frac{dy}{y^2} = \int x^3 \, dx$$

$$-\frac{1}{y} = \frac{x^4}{4} + C$$

$$y = -\frac{1}{\frac{x^4}{4} + C}$$

which can also be written as

$$y = \frac{4}{B - x^4}$$

by setting $B = -4C$.

$$y(x) = \frac{4}{B - x^4}$$

Answer
5. Find the solution $y(x)$ of the initial value problem
\[
\frac{dy}{dx} = \frac{y}{x} - \frac{1}{x}
\]
\[y(1) = 0.\]

Give your final answer in explicit form.

\[
\frac{dy}{dx} = \frac{y}{y - 1}
\]
\[
\frac{dy}{dx} = \frac{dx}{x}
\]
\[
\int \frac{dy}{y - 1} = \int \frac{dx}{x}
\]
\[
\ln |y - 1| = \ln |x| + C
\]
\[
e^{\ln |y - 1|} = e^{\ln |x|} e^{C}
\]
\[
|y - 1| = e^{C} |x|
\]
\[
y - 1 = (\pm e^{C}) x = Kx
\]
\[
y = 1 + Kx
\]
\[
\therefore 0 = y(1) = 1 + K \cdot 1 = 1 + K
\]
\[
K = -1
\]
\[
\therefore y = 1 - x
\]

\[
y(x) = \frac{1 - x}{\text{Answer}}
\]