Chapter 1

Superconnection

1.1 $\mathbb{Z}_2$-graded algebra

Definition 1.1.1. $A$ is a $\mathbb{C}$-algebra, $A$ is $\mathbb{Z}_2$-graded, if there is a splitting: $A = A_+ \oplus A_-$, such that $A_+ A_+ \subset A_+, A_- A_- \subset A_+, A_+ A_- \subset A_-, A_- A_+ \subset A_-$. $A_{\text{even}} = A_+, A_{\text{odd}} = A_-$ are called the even and odd part of $A$. The usual algebra is trivially $\mathbb{Z}_2$-graded with $A = A_+$. 

Definition 1.1.2. for homogeneous elements $a, b \in A_{\pm}$, the supercommutator is 

$$[a, b] = ab - (-1)^{\text{deg}a \cdot \text{deg}b} ba, \quad \text{deg} a = \begin{cases} 0, & a \in A_+ \\ 1, & a \in A_- \end{cases}$$

we extend linearly to $A$. $a, b \in A$ is said to be supercommutative if $[a, b] = 0$. Note that $[A_+, A_+] \subset A_+, [A_-, A_-] \subset A_-, [A_+, A_-] \subset A_-, [A_-, A_+] \subset A_-$. 

Proposition 1.1.1 (Generalized Jacobi Identity). 

$$[a, [b, c]] + (-1)^{\text{deg}a \cdot \text{deg}b + \text{deg}c} [b, [c, a]] + (-1)^{\text{deg}c \cdot \text{deg}a + \text{deg}b} [c, [a, b]] = 0,$$ or equivalently 

$$[a, [b, c]] = [(a, b), c] + (-1)^{\text{deg}a \cdot \text{deg}b} [b, [a, c]] \text{ (derivation property)}$$

Example 1.1.1. 1. $E$ is a finite dimensional vector space. The exterior algebra: $\wedge^* E^* = \wedge^\text{even} E^* \oplus \wedge^\text{odd} E^*$ is a commutative $\mathbb{Z}_2$-graded algebra, since $\omega \wedge \eta = (-1)^{\text{deg} \omega \cdot \text{deg} \eta} \eta \wedge \omega$. 

2. $\tau : E \rightarrow E$ is an endomorphism, s.t. $\tau^2 = id$, then $E$ is $\mathbb{Z}_2$-graded: $E = E_+ \oplus E_-$, $\tau | E_{\pm} = \pm 1$. $\text{End}(E)$ is a $\mathbb{Z}_2$-graded algebra: 

$$\text{End}(E)^{\text{even}} = \{ A \in \text{End}(E) : \tau A = \tau A \}, \text{End}(E)^{\text{odd}} = \{ A \in \text{End}(E) : \tau A = -\tau A \}$$

3. $q$ is a nondegenerate quadratic form on $E$, the Clifford algebra is $\mathbb{Z}_2$-graded: 

$$\alpha : \text{Cl}(E, q) \rightarrow \text{Cl}(E, q)$$ is the involution defined by $\alpha(v) = -v, \forall v \in E$. 

Definition 1.1.3. $A, B$ are $\mathbb{Z}_2$-graded algebra, $A \otimes B$ has a product: 

$$a \otimes b \cdot a' \otimes b' = (-1)^{\text{deg}b \cdot \text{deg}a'} a a' \otimes b b'$$

for homogeneous elements $a, a' \in A, \ b, b' \in B$. It’s well defined and $A \otimes B$ becomes a $\mathbb{Z}_2$-graded algebra, denoted by $A \hat{\otimes} B$: 

$$(A \hat{\otimes} B)_+ = (A_+ \otimes B_+) \oplus (A_- \otimes B_-), \ (A \hat{\otimes} B)_- = (A_+ \otimes B_-) \oplus (A_- \otimes B_+)$$
Remark 1. If $A, B$ are unitary ($A_1 \in A, A_2 \in B$), then $A, B$ are embedded in $A \otimes B$. We can identify $a$ with $a \otimes 1_B$, and $b$ with $1_A \otimes b$, then $a \cdot b = (-1)^{\deg_a \deg_b} a$, so $A$ and $B$ are supercommutative in $A \otimes B$.

Example 1.1.2. $(E_1,q_1), (E_2,q_2)$ are two vector spaces with nondegenerate quadratic forms. Let $E = E_1 \oplus E_2, q = q_1 \oplus q_2 : q((v_1,v_2)) = q_1(v_1) + q_2(v_2)$, then $CL(E,q) = CL(E_1,q_1) \otimes CL(E_2,q_2)$.

Definition 1.1.4. assume $E = E_+ \oplus E_-$ is $\mathbb{Z}_2$-graded defined by involution $\tau$. $\forall A \in End(E)$, the supertrace of $A$ is $Tr_s(A) = Tr(\tau A)$. In blocked matrix, $\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $Tr_s(A) = Tr(a) - Tr(d)$.

Proposition 1.1.2. $\forall A, B \in End(E), Tr_s[A, B] = 0$.

assume $(E, \tau), (F, \sigma)$ are two $\mathbb{Z}_2$-graded vector space, then $E \hat{\otimes} F = (E \otimes F, \tau \otimes \sigma)$ is a $\mathbb{Z}_2$-graded vector space. $\forall A \in End(E), B \in End(F)$, we have

$Tr_s(A \otimes B) = Tr(\tau A \otimes \sigma B) = Tr(\tau A)Tr(\sigma B) = Tr_s(A)Tr_s(B)$.

Definition 1.1.5. $E$ is a finite dimensional, we define the supertrace on the $\mathbb{Z}_2$-graded algebra $\wedge^* E \hat{\otimes} End(E)$(matrix of forms):

$Tr_s : \wedge^* E \hat{\otimes} End(E) \longrightarrow \wedge^* E^*$

$\omega \otimes A \mapsto \omega{Tr_s}(A)$

By the universal property of tensor product, this is well defined. For simplicity, we write $\omega A$ for $\omega \otimes A$. We have

$[\omega A, \omega'A] = (-1)^{\deg A \deg \omega} \omega' [A, A']$

1.2 Superconnection

Definition 1.2.1. $E$ is a finite dimensional $\mathbb{Z}_2$-graded complex vector bundle on the manifold $M$. A superconnection $A$ is an odd differential operator acting on $\Gamma(\wedge^* T^* M \hat{\otimes} E)$, which satisfies the Leibniz rule:

$A(\omega s) = d\omega \cdot S + (-1)^{\deg \omega} \omega \cdot As$

Remark 2. $\wedge^* T^* M \hat{\otimes} E$ is a $\mathbb{Z}_2$-graded vector bundle, $\Gamma(\wedge^* T^* M \hat{\otimes} E)$ is the set of $E$-valued forms which is naturally $\mathbb{Z}_2$-graded. Locally on an open set $U$, we choose a cotangent frame $\{e^i\}$ and a frame of $\{s_\alpha\} = \{s^\alpha_+\} \cup \{s^\alpha_-\}$ of $E = E_+ \oplus E_-$, then locally $\tau \in \Gamma(\wedge^* T^* M \hat{\otimes} E)$ is:

$\tau = \tau_{i_1,...,i_r}^a \wedge \cdots \wedge \tau_{i_1,...,i_r}^a \otimes s_\alpha, \tau_{i_1,...,i_r}^a \in C^\infty(U)$

$A\tau = g_{\beta j_1,...,j_v}^a e^{i_1} \wedge \cdots \wedge e^{i_r} \otimes s_\beta$, $g = D(f), D$ is a matrix of differential operators. $\tau$ is even iff locally it’s the sum of $\omega_{\text{even}} \cdot s^\alpha_+$ and $\omega_{\text{odd}} \cdot s^\alpha_-$.

$\Gamma = \Gamma_{\text{even}} \oplus \Gamma_{\text{odd}} = \Gamma((\wedge^* T^* M \hat{\otimes} E)_{\text{even}}) \oplus \Gamma((\wedge^* T^* M \hat{\otimes} E)_{\text{odd}})$

By definition, $A(\Gamma_{\text{even}}) \subset \Gamma_{\text{odd}}, A(\Gamma_{\text{odd}}) \subset \Gamma_{\text{even}}$.

Example 1.2.1. $\nabla^E$ is a connection preserving the splitting $E = E_+ \oplus E_-$, i.e. $\nabla^E = \nabla^{E_+} \oplus \nabla^{E_-}$, or in blocked matrix: $\nabla^E = \begin{pmatrix} \nabla^{E_+} & 0 \\ 0 & \nabla^{E_-} \end{pmatrix}$

We extend $\nabla^E$ to act on $\Gamma(\wedge^* T^* M \hat{\otimes} E)$ satisfying the Leibniz rule: $\nabla^E(\omega s) = d\omega \cdot s + (-1)^{\deg s} \omega \wedge \nabla^E s$. Then $\nabla^E$ is a superconnection, since it increases the degree of forms by 1, while keeps the degree of sections of the bundle $E$, thus odd.
Proof. Since \( D \in \Gamma(\text{End}(E)^{odd}) \), i.e. \( D \in \text{End}(E_x)^{odd} \), in matrix \( D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} \), then \( \nabla^E + D \) is a superconnection, since \( D \) preserves the degree of forms, and interchanges the even and odd part of \( E \).

In particular, when \( M = \{pt\} \), \( E \) is a \( \mathbb{Z}_2 \)-graded vector space, \( D \in \Gamma(\text{End}(E)^{odd}) \) is a superconnection.

**Proposition 1.2.1.** Any superconnection \( A \) can be written as \( A = \nabla^E + B \), with \( \nabla^E = \nabla^E_+ \oplus \nabla^E_- \), and \( B \in \Gamma(B) \), where \( B = \wedge^* T^* M \hat{\otimes} \text{End}(E) \) is a bundle of algebras.

Proof. Let \( \nabla^E = A \), then \( B \) is an odd operator on \( \Gamma(\wedge^* T^* M \hat{\otimes} E) \). \( \forall f \in C^\infty(M) \), \( s \in \Gamma(\wedge^* T^* M \hat{\otimes} E) \),

\[
B(f(s)) = A(f(s)) - \nabla^E(f(s)) = df \cdot s + fAs - (df \cdot s + f\nabla^E s) = f(A - \nabla^E)s = fBs
\]

so \( B \) is \( C^\infty(M) \) linear, i.e. tensorial.

Let \( Bs_\alpha = \theta_\alpha^a s_\beta \), \( \theta_\alpha^a \in \Gamma(\wedge^* T^* U) \), define \( \tilde{B} = \theta_\alpha^a s_\beta \in \Gamma(\wedge^* T^* M \hat{\otimes} \text{End}(E)) \), where \( \{s_\beta\} \) is the dual basis of \( \{s_\beta\} \), then \( \tilde{B} \) is independent of the chosen frame since \( B \) is tensorial.

Conversely \( \forall \tilde{B} \in \Gamma(\wedge^* T^* M \hat{\otimes} \text{End}(E)) \), \( \tilde{B} \) acts naturally on \( \Gamma(\wedge^* T^* M \hat{\otimes} E) \) : \( (\omega \otimes D) \cdot (\eta \otimes s) = (-1)^{\deg D \cdot \deg \eta} \omega \wedge \eta \otimes Ds \), this action is clearly \( C^\infty \) linear. So we can identify \( \tilde{B} \) and \( B \), and the proposition is proved. \( \square \)

**Remark 3.** if we choose local frame \( \{s^+_\alpha\} \cup \{s^-_\alpha\} \) of \( E \) on \( U \), then \( E \cong \mathbb{C}^n \oplus \mathbb{C}^m \). The de Rham operator \( d \) is a connection on trivial bundles. By the proposition, every connection is locally of the form:

\[
A = d + B, B \in \Gamma(\wedge^* T^* M \hat{\otimes} \text{End}(\mathbb{C}^n \oplus \mathbb{C}^m))_{odd}
\]

### 1.3 Chern-Weil theory for superconnections

**Proposition 1.3.1.** \( A \) is a superconnection, then \( A^2 \in \Gamma(B_{even}) \).

Recall that \( B = \wedge^* T^* M \hat{\otimes} \text{End}(E) \).

Proof. Since \( A \) is odd, \( A^2 \) is an even operator. \( \forall f \in C^\infty(M) \),

\[
A^2(f(s)) = A(df \cdot s + f \cdot As) = (df)s - df \wedge As + df \wedge As + fA^2s = fA^2s
\]

so \( A^2 \) is a tensor. \( \square \)

**Definition 1.3.1.** The curvature of a superconnection \( A \) is \( A^2 \). The Chern character of \( A \) is \( ch(A) = \varphi Tr_s(e^{-A^2}) \in \Gamma(\wedge^* T^* M) \), where

\[
\varphi : \wedge^2_\alpha T^* M \rightarrow \wedge^2_\alpha T^* M = \wedge^* T^* M \otimes \mathbb{C} \\
\alpha \mapsto (2\pi i)^{-\frac{\deg \alpha}{4}} \alpha
\]

is the normalization endomorphism, which makes the definition agree with the Chern character of a complex vector bundle.

**Theorem 1.3.1 (Quillen).** \( ch(A) \) is a closed, even form. \( [ch(A)] = ch(E_+) - ch(E_-) \), \( [ch(A)] \in H^2_{dr} even(M) \) is the cohomological class represented by \( ch(A) \).

Proof. Take a local trivialization \( E = E_+ \oplus E_- \cong \mathbb{C}^n \oplus \mathbb{C}^m \), then \( A = d + B \), where \( B \in \Gamma(\wedge^* T^* M \hat{\otimes} \text{End}(\mathbb{C}^n \oplus \mathbb{C}^m))_{odd} \)

\[
dTr_s(\exp(-A^2)) = \Tr_s(d \exp(-A^2)) = \Tr_s[d, \exp(-A^2)] = \Tr_s[d + B, \exp(-A^2)] = \Tr_s[A, \exp(-A^2)] = 0
\]

In the third equality, we use \( \Tr_s[B, \exp(-A^2)] = 0 \): since \( \exp(-A^2) \) is a tensor, the supertrace of supercommutator vanishes. The last equality follows from the Bianchi identity: \( [A, A^2] = 0 \), which is trivial in this setting.
Remark 4. It’s easy to see that, \( \forall B \in \Gamma(\wedge^* T^* M \otimes \text{End}(E)), [A, B] \in \Gamma(\wedge^* T^* M \otimes \text{End}(E)). \)

Since \( A^2 \) is even, \( \exp(-A^2) \) is even. It is the sum of the form

\[
\text{even form } \otimes \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \text{ odd form } \otimes \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}
\]

but \( T_{rs}(\text{odd form } \otimes \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}) = 0 \), so \( T_{rs}(\exp(-A^2)) \) contains only even forms.

Next we want to prove \( [\text{ch}(A)] \) is independent of the superconnection. Let \( A = \{\text{superconnection}\}, A \) is an affine space. Given \( A_0, A_1 \in A \), let \( A_t = \frac{1}{2}(1-t)A_0 + tA_1 \). Define \( A_t = dt \frac{\partial}{\partial t} + A_t \), then \( A \) is a superconnection on the vector bundle \( \pi^*_M E \to M \times [0, 1] \), where \( \pi_M : M \times [0, 1] \to M \) is the projection. \( A^2 = A_t^2 + dt \frac{\partial}{\partial t} \),

\[
\text{ch}(A) = \varphi Tr_s \exp(-A_t^2 + dt \frac{\partial A_t}{\partial t}) = \text{ch}(A_t) + dt \beta_t
\]

\( \beta_t \) is an odd form on \( M \times [0, 1] \). We know that \( \text{ch}(A) \) is closed, so

\[
0 = d\text{ch}(A) = d_M \text{ch}(A_t) + dt \frac{\partial}{\partial t} \text{ch}(A_t) - dt d_M \beta_t = dt \frac{\partial}{\partial t} \text{ch}(A_t) - dt d_M \beta_t
\]

so \( \frac{\partial}{\partial t} \text{ch}(A_t) = d\beta_t \) and

\[
\text{ch}(A_1) - \text{ch}(A_0) = \int_0^1 \frac{\partial}{\partial t} \text{ch}(A_t) dt = d \int_0^1 \beta_t dt
\]

so \( [\text{ch}(A_0)] = [\text{ch}(A_1)] \).

Thus we can choose any superconnection to compute \( [\text{ch}(A)] \). We let \( A = \nabla^E = \begin{pmatrix} \nabla^E_+ & 0 \\ 0 & \nabla^E_- \end{pmatrix} \), then

\[
\text{ch}(A) = \varphi \text{Tr} \exp(-\nabla^{E,2}) = \varphi \text{Tr} \exp(-\nabla^{E+,-,2}) - \varphi \text{Tr} \exp(-\nabla^{E-,2})
\]

\[
= \text{Tr} \exp(\frac{\sqrt{-1}}{2\pi} \nabla^{E,2}) - \text{Tr} \exp(\frac{\sqrt{-1}}{2\pi} \nabla^{E-,2}) = \text{ch}(E_+) - \text{ch}(E_-)
\]

Example 1.3.1. \( M = \mathbb{C}, E = E_+ \oplus E_- = \mathbb{C} \oplus \mathbb{C}, D_z = \begin{pmatrix} 0 & \bar{z} \\ z & 0 \end{pmatrix}, \nabla^E = d, A_t = \nabla^E + \sqrt{t}d = d + \sqrt{t}d, A_t^2 = d^2 + \sqrt{t}dD + tD^2 = \begin{pmatrix} t|z|^2 & 0 \\ 0 & t|z|^2 \end{pmatrix} + \sqrt{t} \begin{pmatrix} 0 & \bar{z} \\ z & 0 \end{pmatrix}
\]

Since \( \frac{1}{n!} \text{Tr}_s(t|z|^2 + \sqrt{t} \begin{pmatrix} 0 & \bar{z} \\ z & 0 \end{pmatrix})^n = \frac{1}{n!} \frac{1}{2} (t|z|^2)^{n-2} T_{rs} \begin{pmatrix} dzd\bar{z} & 0 \\ 0 & dzd\bar{z} \end{pmatrix} = -t (t|z|^2)^{n-2} \frac{1}{(n-2)!} dzd\bar{z}
\]

\[
T_{rs} \exp(-A_t^2) = \sum_{n=0}^\infty (-1)^n T_{rs}(A_t^2) = -t \exp(-t|z|^2) dzd\bar{z}
\]

\[
\text{ch}(A_t) = \varphi \text{Tr}_s \exp(-A_t^2) = \frac{1}{2\pi i} \cdot (-t) \exp(-t|z|^2) dzd\bar{z} = \frac{t}{\pi} \exp(-t|z|^2) dx dy = P_{\frac{t}{\pi}}(z) dx dy
\]

where \( p_t(x, y) = \frac{1}{4\pi t} \exp(-\frac{|x-y|^2}{4t}) \) is the heat kernel of Laplace \( \Delta = -\left( \frac{d^2}{dx^2} + \frac{d^2}{dy^2} \right) = -4 \frac{d^2}{d^2z} \) on \( \mathbb{C} \). \( \alpha_t = \text{ch}(A_t) \) has the following properties:

\[
\bullet \int_C \alpha_t = 1
\]
\[ H^*_c(\mathbb{R}^n) = H^{n-*}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & * = n \\ 0 & * \neq n \end{cases} \]

The isomorphism \( H^*_c(\mathbb{R}^n) \cong \mathbb{R} \) is given by integration over \( \mathbb{R}^n \).

**Example 1.3.2** (Heat kernel method). \( H_1, H_2 \) are Hilbert spaces, a closed linear operator with dense domain \( P : H_1 \to H_2 \) is Fredholm iff \( \dim ker P < \infty, \dim coker P = \dim H_2/im P < \infty \). We can prove that \( \text{im } P \) is closed, so \( coker P \cong (im P)\perp = ker P^* \), and

\[ \text{Ind}(P) = \dim ker P - \dim coker P = \dim ker P - \dim ker P^* = \dim ker P^* P - \dim ker PP^* \]

\( P^* P \) and \( PP^* \) have the same nonzero eigenvalues, and the corresponding eigenspaces are isomorphic:

\[ E_{P^*P}(\lambda) \xrightarrow{\cong} E_{PP^*}(\lambda) \]

We can obtain the heat equation method:

\[ \text{Ind}(P) = \dim ker P^* P - \dim ker PP^* = \sum_{\lambda} e^{-t\lambda} \dim E_{P^*P}(\lambda) - \sum_{\lambda} e^{-t\lambda} \dim E_{PP^*}(\lambda) \]

\[ = Tr e^{-tP^*P} - Tr e^{-tPP^*} = Tr(e^{-tD^2}) = \text{ch}(D) \]

where \( D = \begin{pmatrix} 0 & P^* \\ P & 0 \end{pmatrix} \) is a superconnection on the infinite dimensional Hilbert bundle \( H = H_1 \oplus H_2 \to \text{pt} \).

We can also use the transgression method to see this:

\[ \frac{\partial}{\partial t} Tr_se^{-tD^2} = -Tr_s(D^2 e^{-tD^2}) = -\frac{1}{2} Tr_s([D,D]e^{-tD^2}) = -\frac{1}{2} Tr_s[D,D e^{-tD^2}] = 0 \]

so \( Tr_se^{-tD^2} \) is a constant. Since \( Tr_se^{-tD^2} \xrightarrow{\text{t} \to \infty} \dim ker P^* P - \dim ker PP^* = \text{Ind}(P) \), the formula follows.

### 1.4 Chern-Simons class

\( E \) is a complex vector bundle, \( \nabla^E, \nabla'^E \) are two connections, \( R^E = \nabla^{E,2}, R'^E = \nabla'^{E,2} \) are their curvatures. \( P \) is an invariant polynomial of \( GL(n, \mathbb{C}) \), i.e. \( P \) satisfies:

\[ P(TAT^{-1}) = P(A), \forall T \in GL(n, \mathbb{C}), A \in M_n \]

\( P(R^E) \) is closed and \([P(R^E)])\] does not depend on \( \nabla^E \), so

\[ P(R^E) - P(R^E) = da, \text{ for some } \alpha \in \Omega^{odd}(M)/d\Omega^{even}(M) \]

We can have a canonical way to construct \( \alpha \), which is called the Chern-Simons form.

As in the proof of Theorem, choose a curve \( C_t = \{ \nabla^E_t : t \in [0,1] \} \) connecting \( \nabla^E \) and \( \nabla'^E \) in \( \mathcal{A} \) which is the affine space of connections. \( \nabla^B = \nabla^E_t + dt \frac{\partial}{\partial t} \) is a connection on the bundle \( \pi_M E \) over \( M \times [0,1] \), where \( \pi_M : M \times [0,1] \to M \) is the projection. The curvature \( R^E = \nabla^{E,2} = \nabla^{E,2}_t + dt \frac{\partial \nabla^E}{\partial t} \).
\[ P : M_n \cong \mathbb{C}^{n^2} \to \mathbb{C} \text{ is a polynomial of } n^2 \text{ variables, we have the Taylor expansion:} \]
\[ P(x + y) = \sum_{|\alpha| \leq \deg P} \frac{\partial^\alpha P(x)}{\alpha!} y^\alpha, \forall x, y \in \mathbb{C}^{n^2} \]

Note that each term of \( R^E \) is a 2-form, which commutes with other forms. So we use the above formula to have:

\[ P(R^E) = P(R^E_t + dt \frac{\partial}{\partial t} \nabla^E_t) = P(R^E_t) + dt < P'(R^E_t), \frac{\partial}{\partial t} \nabla^E_t > \]

where \( P'(x) : \mathbb{C}^{n^2} \to \mathbb{C} \) is the total derivative at \( x \in \mathbb{C}^{n^2} \). We know that \( P(R^E) \) is closed, so:

\[ 0 = d_M \times [0,1] P(R^E) = d_M P(R^E_t) + dt \frac{\partial}{\partial t} P(R^E_t) - dt d_M < P'(R^E_t), \frac{\partial}{\partial t} \nabla^E_t > \]

so \( \frac{\partial}{\partial t} P(R^E_t) = d < P'(R^E_t), \frac{\partial}{\partial t} \nabla^E_t > \). Define

\[ \tilde{P} = \int_0^1 < P'(R^E_t), \frac{\partial}{\partial t} \nabla^E_t > dt \]

then \( d\tilde{P} = P(R^E) - P(R^E) \).

This is a case of integration along the fibre, see appendix. We write:

\[ \tilde{P} = \int_{C_t} P(R^E_t) = \int_{[0,1]} P(R^E) \]

**Proposition 1.4.1.** The class of \( \tilde{P} \) in \( \Omega^{odd}/d\Omega^{even} \) does not depend on the path connecting \( \nabla^E \) and \( \nabla^iE \).

**Proof.** If \( C'_t = \{\nabla^iE \} \) is another path, let \( \nabla^E_{s,t} = (1-s)\nabla^E_t + s\nabla^iE \), define \( \nabla^E = ds \frac{\partial}{\partial s} + dt \frac{\partial}{\partial t} + \nabla^E_{s,t} \). then \( \nabla^E \) is a connection on \( E = p_1^*E, p_1 : M \times [0,1]^2 \to M \) is the projection. The curvature:

\[ R^E = \nabla^E,2 = \nabla^E_{s,t} + dt \frac{\partial}{\partial t} \nabla^E_{s,t} + ds \frac{\partial}{\partial s} \nabla^E_{s,t} = \nabla^E_{s,t} + dt((1-s) \frac{\partial}{\partial t} \nabla^E_t + s \frac{\partial}{\partial t} \nabla^iE_t) + ds(\nabla^iE - \nabla^E_t) \]

\[ R^E \|_{s=0} = \nabla^E,2 + dt \frac{\partial}{\partial t} \nabla^E_t = \nabla^E,2, R^E \|_{s=1} = \nabla^iE,2 + dt \frac{\partial}{\partial t} \nabla^iE_t = \nabla^iE,2 \]

\[ R^E \|_{t=0} = \nabla^E,2, R^E \|_{t=1} = \nabla^iE,2 \]

Note that \( R^E \|_{t=0}, R^E \|_{t=1} \) have no ds term. Using the Stoke’s Formula, we have:

\[ d \int_{[0,1]^2} P(R^E) = \int_{[0,1]^2} dP(R^E) - (\int_{I_1} P(R^E \|_{s=0}) - \int_{I_1} P(R^E \|_{s=1})) \]

\[ = -\int_{[0,1]} P(R^E) + \int_{[0,1]} P(R^E) = \tilde{P} - \tilde{P} \]

\[ \]
1. \( d \tilde{P}(\nabla^E, \nabla'^E) = P(R^E) - P(R^E) \)
2. \( \tilde{P}(\nabla^E, \nabla^E) = 0 \) Just choose the constant path: \( \nabla^E_t \equiv \nabla^E \).
3. \( \tilde{P}(\nabla^E, \nabla'^E) \) is functorial: if \( f : N \to M \) is a differentiable mapping, then
   \[
   \tilde{P}(f^* \nabla^E, f^* \nabla'^E) = f^* \tilde{P}(\nabla^E, \nabla'^E)
   \]
   where \( f^* \nabla^E, f^* \nabla'^E \) is the induced connection on \( F = f^* E \).

This is because: Choose a curve \( \nabla^E_t \) connecting \( \nabla^E \) and \( \nabla'^E = f^* \nabla^E \) is a curve connecting \( \nabla^E = \nabla^E \) and \( \nabla'^E = f^* \nabla^E \). Let \( E = \pi_M^* E, \nabla^E = \nabla^E_t + dt \frac{\partial}{\partial t}; F = \pi_N^* F, \nabla^F = \nabla^E_t + dt \frac{\partial}{\partial t} = (f \times \text{id})^* \nabla^E \).

\[
\tilde{P}(f^* \nabla^E, f^* \nabla'^E) = \int_{[0,1]} P(R^E) = \int_{[0,1]} P((f \times \text{id})^* R^E) = f^* \int_{[0,1]} P(R^E) = f^* \tilde{P}(\nabla^E, \nabla'^E)
\]

In fact, we have:

**Proposition 1.4.2.** The class \( \tilde{P}(\nabla^E, \nabla'^E) \) is uniquely defined by properties 1-3

**Proof.** We need to check the uniqueness. Assume \( \tilde{P} \) satisfies the 3 properties. Choose any path \( \nabla^E_t \) connecting \( \nabla^E \), \( \nabla'^E \). Let \( \nabla^E = \nabla^E + dt \frac{\partial}{\partial t}, \nabla'^E = \nabla^E + dt \frac{\partial}{\partial t} \) be connections on \( E = \pi_M^* E \). By property 1, we have \( P(R^E) - P(R'^E) = d\tilde{P}(\nabla^E_0, \nabla'^E) \). Integrate this equality along the fibre, and use the Stoke’s Formula:

\[
\int_{[0,1]} P(R^E) = \int_{[0,1]} P(R'^E) + \int_{[0,1]} d\tilde{P}(\nabla^E_0, \nabla'^E)
\]

\[
= -d \int_{[0,1]} \tilde{P}(\nabla^E_0, \nabla'^E) + \tilde{P}(\nabla^E_0, \nabla^E)|_{t=1} - \tilde{P}(\nabla^E_0, \nabla'^E)|_{t=0}
\]

Since \( \nabla^E_0|_{t=1} = \nabla^E, \nabla^E|_{t=1} = \nabla'^E, \) by functoriality, \( \tilde{P}(\nabla^E_0, \nabla^E)|_{t=1} = \tilde{P}(\nabla^E, \nabla'^E) \). Also \( \tilde{P}(\nabla^E_0, \nabla'^E)|_{t=0} = \tilde{P}(\nabla^E, \nabla'^E) = 0 \) by property 2. So \( \tilde{P}(\nabla^E, \nabla'^E) - \int_{[0,1]} P(R^E) \) is exact, that is what we require.

**Example 1.4.1.** \( \lambda \) is a line bundle, \( c_1(\lambda) \) is the first Chern class, then \( \tilde{c}_1(\nabla^E, \nabla'^E) = -\frac{\nabla'^E - \nabla^E}{2\pi i} \) is the Chern-Simon class. This is because \( d(-\frac{\nabla'^E - \nabla^E}{2\pi i}) = \frac{i}{2\pi} r'^E - \frac{i}{2\pi} r^E = c_1(\nabla'^E) - c_1(\nabla^E) \), and the other two properties are obvious.

Since \{superconnections\} is an affine space as \{connections\}, everything we’ve done can be carried out on superconnections, so we have:

**Proposition 1.4.3.** The whole theory of Chern-Simon class extends to superconnections

**Proposition 1.4.4.** Let \( A_t \) be a curve of superconnections, then

\[
\frac{\partial}{\partial t} \text{Tr}_s(\exp(-A_t^2)) = -d\text{Tr}_s(\frac{\partial A_t}{\partial t} \exp(-A_t^2))
\]

**proof 1.** Let \( \lambda = A_t + dt \frac{\partial}{\partial t} \) be a superconnection on \( \pi_M^* E \). By simple computation as example 1.3.1, we see that

\[
\text{Tr}_s(\exp(-\lambda^2)) = \text{Tr}_s(\exp(-A_t^2)) - dt \text{Tr}_s(\frac{\partial A_t}{\partial t} \exp(-A_t^2))
\]

We know that \( \text{Tr}_s(\exp(-\lambda^2)) \) is a closed even form on \( M \times \mathbb{R} \), so

\[
\frac{\partial}{\partial t} \text{Tr}_s(\exp(-A_t^2)) = -d\text{Tr}_s(\frac{\partial A_t}{\partial t} \exp(-A_t^2))
\]

\[\square\]
\[ \frac{\partial}{\partial t} \text{Tr}_s \exp(-A_t^2) = -\text{Tr}_s \left( \frac{\partial A_t^2}{\partial t} \exp(-A_t^2) \right) = -\text{Tr}_s \left( [A_t, \frac{\partial A_t}{\partial t} \exp(-A_t^2)] \right) = -\text{Tr}_s \left( [A_t, \frac{\partial A_t}{\partial t} \exp(-A_t^2)] \right) = -d\text{Tr}_s \left( \frac{\partial A_t}{\partial t} \exp(-A_t^2) \right) \]

We used identities:

- \( \frac{\partial A_t^2}{\partial t} = \frac{\partial A_t}{\partial t} A_t + A_t \frac{\partial A_t}{\partial t} = [A_t, \frac{\partial A_t}{\partial t}] \) (note both \( A_t \) and \( \frac{\partial A_t}{\partial t} \) are odd)
- \( d\text{Tr}_s(B) = \text{Tr}_s[A, B] \), for \( A \) a superconnection and \( B \in \Gamma(B) \) (see the proof of theorem 1.3.1)

\[
\text{When } A_t = \nabla^E + \sqrt{t}V, \nabla^E = \nabla^{E+} \oplus \nabla^{E-}, V = \left( \begin{array}{cc} 0 & V_+ \\ V_- & 0 \end{array} \right) \in \Gamma(\text{End}(E)^{\text{odd}}),
\]

\[ \frac{\partial}{\partial t} \text{Tr}_s \exp(-A_t^2) = -d\text{Tr}_s \left( \frac{V}{2\sqrt{t}} \exp(-A_t^2) \right) \]

We want to integrate this equality on \([0, \infty)\). Now assume \( g^E = g^{E+} \oplus g^{E-} \) is an Hermitian metric on \( E \), \( \nabla^E \) is a unitary connection, and \( V \) is self-adjoint, so \( V_- = V_+^* \).

**Proposition 1.4.5.** \( \alpha_t = \text{ch}(A_t) \) is a real form.

**Proof.** We define an adjoint operator * on \( B = \Lambda^* T^* M \otimes \text{End}(E) \):

1. \( \forall \omega \in \wedge_1^2 T^* M, \omega^* = -\overline{\omega} \)

2. \( \forall A \in \text{End}(E), A^* \) is the adjoint of \( A \)

3. \( \forall f, g \in B, (f \cdot g)^* = g^* \cdot f^* \)

This is well defined. Now \( A_t^2 = \nabla^{E,2} + \sqrt{t} \nabla^E V + tV^2 \). \((tV^2)^* = tV^2 \), since \( V \) is self-adjoint. \( \forall X \in TM, \nabla_X V \) is self-adjoint, so \( (\nabla^E V)^* = (e^i \nabla^E_{e_i} V)^* = \nabla^E_{e_i} V \cdot (-e^i) = e^i \nabla^E V = \nabla^E V \).

Since \( \nabla^E \) is a unitary connection, \( R^E(X, Y) = \nabla^{E,2}(X, Y) \) is skew-Hermitian w.r.t. \( g^E, \forall X, Y \in TM \), so

\[
(\nabla^{E,2})^* = (e^i \wedge e^j R^E(e_i, e_j))^* = -R^E(e_i, e_j)(-e^i) \cdot (-e^j) = e^i \wedge e^j R^E(e_i, e_j) = \nabla^{E,2}
\]

So we see \((A_t^2)^* = A_t^2\), also \((\exp(-A_t^2))^* = \exp(-A_t^2)\). Note that \( \forall \omega A \in B_{\text{even}}, \)

\[
(\omega A)^* = A^* \omega^* = A^* (-1)^{\deg \omega} \cdot (-1)^{\deg \omega (\deg \omega - 1)} \overline{\omega} = (-1)^{\deg \omega (\deg \omega - 1)} \overline{\omega} A^* 
\]

So

\[
\text{Tr}_s(\omega A) = \text{Tr}_s(\overline{\omega} A^*) = (-1)^{\deg \omega (\deg \omega - 1)} \frac{1}{4} \text{Tr}_s((\omega A)^*) = \begin{cases} \text{Tr}_s((\omega A)^*) & \deg \omega \equiv 0 \mod 4 \\ -\text{Tr}_s((\omega A)^*) & \deg \omega \equiv 2 \mod 4 \\ 0 & \text{otherwise} \end{cases}
\]

So \( (\text{Tr}_s \exp(-A_t^2))^{(4k)} \) is real, and \( (\text{Tr}_s \exp(-A_t^2))^{(4k+2)} \) is purely imaginary, then it’s clear \( \varphi \text{Tr}_s \exp(-A_t^2) = \text{ch}(A_t) \) is real.

**Remark 5.** Similarly, \( \forall \omega A \in B_{\text{odd}}, (\omega A)^* = (-1)^{\deg \omega (\deg \omega - 1)} \overline{\omega} A^* \), so \( \forall B \in B_{\text{odd}} \) satisfying \( B^* = B \), we have \( (\text{Tr}_s(B))^{(4k+1)} \) is purely imaginary, and \( \text{Tr}_s(B)^{(4k+3)} \) is real, then \( \frac{1}{\sqrt{2\pi t}} \varphi \text{Tr}_s(B) \) is real.

**Proposition 1.4.6.** Assume \( \ker V \) has locally constant dimensions, then \( \ker V \) is a smooth subbundle of \( E \).
Proof. Since \( \ker V \) has locally constant dimensions, \( \forall x \in M, \exists \varepsilon > 0 \), and a neighborhood \( U \ni x \), s.t. \( \forall y \in U, V_y : E_y \to E_y \) has no nonzero eigenvalues in the disk \( B(0, \varepsilon) = \{ z \in \mathbb{C} : |z| \leq \varepsilon \} \). Let

\[
P^\ker V = \frac{1}{2\pi i} \int_{\partial B(0, \varepsilon)} \frac{d\lambda}{\lambda - V}
\]

then \( P^\ker V : E \to E \) is a smooth projection onto \( \ker V \). We have the direct sum:

\[
E = P^\ker V(E) \oplus (1 - P^\ker V)(E) = \ker V \oplus (\ker V)^\perp = \ker V \oplus \text{im} V
\]

\( \square \)

Since \( \ker V = \ker V_+ \oplus \ker V_- \), and \( \dim \ker V = \dim(\text{im} V_+) = \dim E_- - (\dim E_+ - \dim \ker V_+) \),

\[
\dim \ker V_+ + \dim \ker V_- = \dim \ker V, \quad \dim \ker V_+ - \dim \ker V = \dim E_+ - \dim E_-
\]

So \( \dim \ker V_+ \), \( \dim \ker V_- \) is locally constant, by the proposition \( \ker V_\pm \) is subbundles of \( E_\pm \).

Let \( g^E \) be the metric on \( \ker V \) induced by \( g^E \), and \( \nabla^\ker V = P^\ker V \nabla^E \) be the induced connection on \( \ker V \) by orthogonal projection, then

\[
\nabla^\ker V = P^\ker V(\nabla^E_+ \oplus \nabla^E_-) = P^\ker V_+ \nabla^E_+ \oplus P^\ker V_- \nabla^E_- = \nabla^\ker V_+ \oplus \nabla^\ker V_-
\]

**Proposition 1.4.7.** \( \text{ch}(E) = \text{ch}(\ker V) \) in \( H^*(M) \), where \( \text{ch}(E) = \text{ch}(E_+) - \text{ch}(E_-) \), \( \text{ch}(\ker V) = \text{ch}(\ker V_+) - \text{ch}(\ker V_-) \).

**Proof.** We have an exact sequence of vector bundles:

\[
0 \to \ker V_+ \to E_+ \to \ker V_-= \ker V_+ \oplus (\ker V_+)^\perp \to \ker V_- \to 0
\]

since we can take any connection to compute \( \text{ch}(E) \), we let

\[
\nabla^E_+ = \nabla^\ker V_+ \oplus (\nabla^\ker V_+)^\perp, \quad \nabla^E_- = (\nabla^\ker V_-)^\perp \oplus \nabla^\ker V_-
\]

where \( (\nabla^\ker V_+)^\perp = V_+^* \nabla(\ker V_-)^\perp \), then

\[
\text{ch}(E) = \text{ch}(E_+) - \text{ch}(E_-) = [\text{ch}(\nabla^E_+)] - [\text{ch}(\nabla^E_-)]
\]

\[
= [\text{ch}(\nabla^\ker V_+)] - [\text{ch}(\nabla^\ker V_-)]
\]

\( \square \)

**Theorem 1.4.1.** When \( t \to +\infty \), we have asymptotic formula:

\[
\text{ch}(A_t) = \text{ch}(\nabla^\ker V) + O\left(\frac{1}{\sqrt{t}}\right)
\]

uniformly on compact sets.

**Proof.** We first compute \( \nabla^\ker V : 2 \). Consider the orthogonal splitting \( E = \ker V \oplus (\ker V)^\perp \), \( \nabla^E = \nabla^{\text{split}} + A \), where \( \nabla^{\text{split}} = \nabla^\ker V \oplus (\nabla^\ker V)^\perp \). In blocked matrix with respect to the new splitting, because \( \nabla^E \) is unitary,

\[
\nabla^E = \begin{pmatrix} \nabla^\ker V & 0 \\ 0 & (\nabla^\ker V)^\perp \end{pmatrix} + \begin{pmatrix} 0 & -B^* \\ B & 0 \end{pmatrix}
\]

Because \( \forall s \in \Gamma(\ker V) \), \( X \in TM \),

\[
(\nabla^E_X s) = \nabla^E_X (Vs) - V(\nabla^E_X s) = -V(\nabla^E_X s) \in \text{im} V
\]
\( \nabla^E V \) is a 1-form with value in \( \text{End}(E) \), which maps \( \ker V \) into \( (\ker V)^\perp = \text{im} V \), and 
\[
P(\ker V)^\perp \nabla^E_X s = -V^{-1}(\nabla^E_X V)s, \text{ where } V : (\ker V)^\perp \to \text{im} V = (\ker V)^\perp \text{ is an isomorphism.}
\]
So
\[
B(X) = P(\ker V)^\perp \nabla^E_X \cdot P\ker V = -V^{-1}(\nabla^E_X V)P\ker V, \quad B(X)^* = -P\ker V \cdot \nabla^E_X V \cdot V^{-1}
\]
or in our sign rule
\[
B = V^{-1} \cdot (\nabla^E V) \cdot P\ker V, \quad B^* = -P\ker V \nabla^E V \cdot V^{-1}
\]

so
\[
A \triangleq \left( \begin{array}{cc} 0 & -B^* \\ B & 0 \end{array} \right) = \left( \begin{array}{cc} 0 & P\ker V \nabla^E V \cdot V^{-1} \\ V^{-1} \nabla^E V & 0 \end{array} \right)
\]

\( \nabla^{E,2} = \nabla^{\text{split},2} + \nabla^{\text{split}} A + A^2 \), since \( \nabla^{\text{split}} A \) interchanges \( \ker V \) and \( (\ker V)^\perp \),
\[
\nabla^{\ker V,2} = P\ker V \nabla^{E,2} P\ker V - P\ker V A^2 P\ker V = P\ker V (\nabla^{E,2} - \nabla^E V \cdot V^{-1} \cdot \nabla^E V) P\ker V
\]

we next take the limit. \( A_t^2 = tV^2 + \sqrt{t} \nabla^E V + \nabla^{E,2} \in \Gamma(B_{\text{even}}) \), \( \sqrt{t} \nabla^E V + \nabla^{E,2} \) is nilpotent in \( B \) because they contain Grassmannian variables. So the spectrum of \( A_t^2 \) in \( B \) is the same as \( tV^2 \).

Since \( tV^2 \) is positive, \( \text{Sp}(A_t^2) = \text{Sp}(tV^2) \subset \mathbb{R}_+ \). We choose a contour \( C \) as in the figure and use the Cauchy integral formula:
\[
\exp(-A_t^2) = \frac{1}{2\pi i} \int_C \exp(-\lambda) d\lambda \quad \frac{1}{\lambda - A_t^2}
\]

We can compute \( (\lambda - A_t^2)^{-1} \) by the following

**Lemma 1.** \( M = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \) is a blocked matrix, then
\[
(\lambda I - M)^{-1} = \left( \begin{array}{cc} \alpha & \alpha B(\lambda - D)^{-1} \\ (\lambda - D)^{-1} C\alpha & (\lambda - D)^{-1}(1 + C\alpha B(\lambda - D)^{-1}) \end{array} \right)
\]
where
\[
\alpha = (\lambda - A - B(\lambda - D)^{-1}C)^{-1}
\]
the proof is just computation.

\[
A_t^2 = \left( \begin{array}{cc} 0 & 0 \\ 0 & tV^2 \end{array} \right) + \sqrt{t} \left( P(\ker V)^\perp \nabla^E V P(\ker V)^\perp + P(\ker V)^\perp \nabla^E V P(\ker V)^\perp \right) + \left( \begin{array}{cc} P(\ker V)^\perp \nabla^E,2 P\ker V & P(\ker V)^\perp \nabla^E,2 P(\ker V)^\perp \\ P(\ker V)^\perp \nabla^E,2 P\ker V & P(\ker V)^\perp \nabla^E,2 P(\ker V)^\perp \end{array} \right)
\]

Using the lemma, we have
\[
\alpha_t = [\lambda - P\ker V \nabla^{E,2} P\ker V - (\sqrt{t} P\ker V \nabla^E V P(\ker V)^\perp + O(1))(\lambda - tV^2 - O(\sqrt{t}))^{-1}]
\]
\[
= (\lambda - P\ker V (\nabla^{E,2} - \nabla^E V V^{-2} \nabla^E V) P\ker V)^{-1} + O\left( \frac{1}{\sqrt{t}} \right)
\]
\[
= (\lambda - \nabla^{\ker V,2})^{-1} + O\left( \frac{1}{\sqrt{t}} \right)
\]
and estimates for other blocks:

\[
\begin{array}{cccc}
\alpha & B & (\lambda - D)^{-1} & = O\left(\frac{1}{\sqrt{t}}\right), \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & \sqrt{t} & t^{-1} & t^{-1}
\end{array}
\]

\[
(\lambda - D)^{-1} C \alpha = O\left(\frac{1}{\sqrt{t}}\right)
\]

So \((\lambda - A^2)^{-1} = \begin{pmatrix} (\lambda - \nabla^{kerV,2})^{-1} & 0 \\ 0 & 0 \end{pmatrix} + O\left(\frac{1}{\sqrt{t}}\right), \) and

\[
\exp(-A^2_t) = \frac{1}{2\pi i} \int_C \frac{\exp(-\lambda)}{\lambda - A^2_t} d\lambda = \frac{1}{2\pi i} \int_C \frac{\exp(-\lambda)}{\lambda - \nabla^{kerV,2}} P^{kerV} + O\left(\frac{1}{\sqrt{t}}\right)
\]

\[
ch(A_t) = ch(\nabla^{kerV}) + O\left(\frac{1}{\sqrt{t}}\right)
\]

as \(t \to +\infty,\) uniformly on compact sets of \(M.\)

\[\square\]

**Proposition 1.4.8.**

\[Tr_s(\sqrt{t}V \exp(-A^2_t)) = O\left(\frac{1}{t}\right), \text{ as } t \to +\infty\]

**Proof.** Consider the projection \(\pi : M \times (0, \infty) \to M, \nabla^{E} = \nabla^{E} + da \frac{\partial}{\partial a}\) is a connection on \(E = \pi^* E.\) Let \(A_t = \nabla^{E} + \sqrt{t}aV\) be a superconnection.

\[A^2_t = \nabla^{E,2} + ta^2 V^2 + \sqrt{t}a \nabla^{E} V + da \sqrt{t}V\]

So

\[Tr_s(\exp(-A^2_t)) = Tr_s(\exp(-A_t(a)^2)) - da Tr_s(\sqrt{t}V \exp(-A_t(a)^2))\]

where \(A_t(a) = \nabla^{E} + \sqrt{t}aV.\)

Note \(ker(aV) = \pi^* ker V,\) by the above theorem, \(Tr_s(\exp(-A_t(a)^2)) = Tr_s(\exp(-\nabla^{kerV,2}^a)) + O\left(\frac{1}{\sqrt{t}}\right),\)

as \(t \to +\infty,\) so

\[Tr_s(\exp(-A^2_t)) = Tr_s(\exp(-\nabla^{kerV,2})) - da Tr_s(\sqrt{t}V \exp(-A_t(a)^2)) + O\left(\frac{1}{\sqrt{t}}\right)
\]

On the other hand, by the theorem again,

\[Tr_s(\exp(-A^2_t)) = Tr_s(\exp(-\nabla^{ker(aV),2})) + O\left(\frac{1}{\sqrt{t}}\right) = Tr_s(\exp(-\nabla^{kerV,2})) + O\left(\frac{1}{\sqrt{t}}\right)\]

We used the fact: \(\nabla^{ker(aV)} = \pi^* \nabla^{ker V} + da \frac{\partial}{\partial a},\) and so \(\nabla^{ker(aV),2} = \pi^* \nabla^{kerV,2}.\) Comparing the above two equation, and note that \(A_t(1) = A_t,\) we see that

\[Tr_s(\sqrt{t}V \exp(-A^2_t)) = O\left(\frac{1}{\sqrt{t}}\right)\]

\[\square\]

By the above proposition, we have

\[
\frac{\partial}{\partial t} Tr_s(\exp(-A^2_t)) = -dt \exp\left(\frac{V}{2\sqrt{t}} \exp(-A^2_t)\right) = \begin{cases} O\left(\frac{1}{t^2}\right) & \text{as } t \to +\infty \\ O\left(\frac{1}{\sqrt{t}}\right) & \text{as } t \to 0 \end{cases}
\]

So we can integrate it from 0 to \(+\infty.\)

**Definition 1.4.1.** \(\beta = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \varphi Tr_s\left(\frac{V}{2\sqrt{t}} \exp(-A^2_t)\right) dt\)
Theorem 1.4.2. \( \beta \) is an odd real form and
\[
d\beta = ch(\nabla^E) - ch(\nabla^{kerV})
\]

Proof. By the Remark following proposition 1.4.5, \( \beta \) is an odd real form. By theorem 1.4.1 and proposition 1.4.4
\[
Tr_s(\exp(-\nabla^{kerV,2})) - Tr_s(\exp(-\nabla^{E,2})) = \int_0^\infty \frac{d}{dt} Tr_s(\exp(-A_t^2))dt \\
= -d \int_0^\infty Tr_s(\frac{V}{2\sqrt{t}} \exp(-A_t^2))dt
\]
Since \( \varphi \circ d = \frac{1}{\sqrt{2\pi}} d \circ \varphi \),
\[
\varphi \circ d = \frac{1}{\sqrt{2\pi}} d \circ \varphi \cdot Tr_s(\exp(-\nabla^{E,2})) - \varphi \circ d
\]
\[
\varphi \circ d = \frac{1}{\sqrt{2\pi}} \int_0^\infty \varphi \circ d(\frac{V}{2\sqrt{t}} \exp(-A_t^2))dt
\]
i.e. \( ch(\nabla^E) - ch(\nabla^{kerV}) = d\beta \). \( \square \)

We will show \( \beta \) is a Chern-Simon class. First we generalize the definition.

Theorem 1.4.3. Given an exact sequence of vector bundles:
\[
E : 0 \to E_1 \xrightarrow{v_1} E_2 \xrightarrow{v_2} \cdots \xrightarrow{v_n} E_n \to 0
\]
and connections \( \nabla^E = \{ \nabla^E_i \} \) on \( E = \{ E_i, v_i \} \), there is a unique way to associate a class \( \tilde{ch}(\nabla^E) \) \( \in \Omega^{odd}(M)/d\Omega^{even} \), which satisfies 3 properties:

1. \( d\tilde{ch}(E) = ch(\nabla^{E_1}) - ch(\nabla^{E_2}) + \cdots + (-1)^{n-1}ch(\nabla^{E_n}) =: ch(\nabla^E) \)

2. If the sequence \( (E, \nabla^E) \) splits, \( \tilde{ch}(\nabla^E) = 0 \). “(E, \nabla^E) splits” means:
\[
E : 0 \to E_1 \xrightarrow{v_1} E_2' \oplus E_2'' \xrightarrow{v_2} \cdots \xrightarrow{v_n} E_n \to 0
\]
and \( \nabla^{E_i} = \nabla^{E_i'} \oplus \nabla^{E_i''}, \nabla^{E_i'} = v_i^* \nabla^{E_{i+1}'}, \) where \( E_i' = Im v_{i-1} = ker v_i, E_i'' \cong E_i/E_i' \).

3. \( \tilde{ch}(\nabla^E) \) is functorial.

Proof. Fix a splitting: \( E_i = E_i' \oplus E_i'', \nabla^{E_i, split} = \nabla^{E_i'} \oplus \nabla^{E_i''}, \nabla^{E_i'} = v_i^* \nabla^{E_{i+1}} \). Let
\[
\alpha(\nabla^E) = \sum_{i=1}^n (-1)^{i-1} \tilde{ch}(\nabla^{E_i, split}, \nabla^{E_i})
\]
where \( \tilde{ch}(\nabla^E, \nabla'^E) \) is the Chern-Simon class defined before(see proposition 1.4.2). We’ll show \( \alpha(\nabla^E) \) satisfies the 3 properties.

1. Since \( \sum_{i=1}^n (-1)^{i-1} ch(\nabla^{E_i, split}) = 0 \),
\[
d\alpha(\nabla^E) = \sum_{i=1}^n (-1)^{i-1} d\tilde{ch}(\nabla^{E_i, split}, \nabla^{E_i}) = \sum_{i=1}^n (-1)^{i-1}(ch(\nabla^{E_i}) - ch(\nabla^{E_i, split})) = \sum_{i=1}^n (-1)^{i-1} ch(\nabla^{E_i}) = ch(\nabla^E)
\]

2. First consider another splitting of the kind: \( E_i = E_i' \oplus \tilde{E}_i'' \), \( \nabla^{E_i} = \nabla^{E_i'} \oplus \tilde{\nabla}^{E_i''}, \nabla^{E_i'} = v_i^* \nabla^{E_{i+1}'}, \) i.e. the bundles have a new splitting \( E_i = E_i' \oplus E_i'' \), but we use the fixed \( \tilde{\nabla}^{E_i} \). They together determine the new splitting connections \( \nabla^{E_i} = \nabla^{E_i'} \oplus \tilde{\nabla}^{E_i''} \).
Claim 1. \(\tilde{ch}(\nabla^{E_i,split}, \nabla^{E_i}) = 0\).

proof of the Claim. Let \(\tilde{P}^{E_i'}\) and \(\tilde{P}^{E_i''}\) be the projection w.r.t this new splitting, we have isomorphisms

\[
(E_i'', \nabla^{E_i''}) \xrightarrow{\tilde{P}^{E_i''}|_{E_i''}} E_i'' \xrightarrow{\sim} \tilde{E}_i'' \xrightarrow{\sim} (\tilde{E}_i'', \nabla^{\tilde{E}_i''})
\]

So \(\nabla^{E_i} = \nabla^{E_i'} \oplus (\nabla^{E_i'')}^* (\nabla^{E_i''})\). \(\forall s \in E_i, s = u + v = \tilde{u} + \tilde{v}\) is decompositions w.r.t the two splittings, we calculate

\[
P^{E_i'}(\nabla^{E_i'} \oplus (\nabla^{E_i'')}^*) (\nabla^{E_i''}) P^{E_i'} s = P^{E_i'} \nabla^{E_i'} \tilde{P}^{E_i'} P^{E_i'} s + P^{E_i'}(P^{E_i''})^{-1} \nabla^{E_i'} P^{E_i'} P^{E_i'} s
\]

\[
= \nabla^{E_i'} P^{E_i'} s
\]

\[
P^{E_i'}(\nabla^{E_i'} \oplus (\nabla^{E_i'')}^*) (\nabla^{E_i''}) P^{E_i'} s = P^{E_i'} \nabla^{E_i'} \tilde{P}^{E_i'} P^{E_i'} s + P^{E_i'}(P^{E_i''})^{-1} \nabla^{E_i'} P^{E_i'} P^{E_i'} s
\]

\[
= \nabla^{E_i'} P^{E_i'} s
\]

\[
P^{E_i'}(\nabla^{E_i'} \oplus (\nabla^{E_i'')}^*) (\nabla^{E_i''}) P^{E_i'} s = P^{E_i'} \nabla^{E_i'} \tilde{P}^{E_i'} P^{E_i'} s + P^{E_i'}(P^{E_i''})^{-1} \nabla^{E_i'} P^{E_i'} P^{E_i'} s
\]

\[
= \nabla^{E_i'} P^{E_i'} s = 0
\]

In the above calculation, we use some identities, like

\[
\tilde{P}^{E_i'} P^{E_i'} = P^{E_i'}, \tilde{P}^{E_i''} P^{E_i'} = 0, \tilde{P}^{E_i'} P^{E_i''} = P^{E_i''}, P^{E_i'}(P^{E_i''})^{-1} = P^{E_i'} \tilde{P}^{E_i''} = 0
\]

So with respect to the initial splitting \(E_i = E_i' \oplus E_i''\), we have

\[
\nabla^{E_i} = \begin{pmatrix} \nabla^{E_i'} & 0 \\ 0 & \nabla^{E_i''} \end{pmatrix} + \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} = \nabla^{E_i,split} + D
\]

Let \(\nabla^{E_i}_{t} = \nabla^{E_i,split} + tD\) be a curve connecting \(\nabla^{E_i,split}\) and \(\nabla^{E_i}\). It’s easy to see that

\[
ch(\nabla^{E_i}_{t}) = ch(\nabla^{E_i,split})
\]

So \(\tilde{ch}(\nabla^{E_i,split}, \nabla^{E_i}) = 0\). \(\square\)

Now we consider the general splitting \(E_i = E_i' \oplus \tilde{E}_i'\), \(\nabla^{E_i} = \nabla^{E_i'} \oplus \nabla^{\tilde{E}_i'}\), \(\nabla^{E_i''} = v_i^* \nabla^{E_i'+1}\). From the definition, we can see that

\[
\tilde{ch}(\nabla^{E_i,split}, \nabla^{E_i}) = \tilde{ch}(\nabla^{E_i,split}, \nabla^{E_i}) + \tilde{ch}(\nabla^{E_i'}, \nabla^{E_i'})
\]

By the above construction, we have

\[
\tilde{ch}(\nabla^{E_i'}, \nabla^{E_i'}) = \tilde{ch}(\nabla^{E_i'} \oplus \nabla^{E_i''}, \nabla^{E_i'} \oplus \nabla^{E_i''})
\]

\[
= ch(\nabla^{E_i'}, \nabla^{E_i'}) + ch(v_i^* \nabla^{E_i'+1}, v_i^* \nabla^{E_i'+1})
\]

\[
= \tilde{ch}(\nabla^{E_i'}, \nabla^{E_i'}) + \tilde{ch}(\nabla^{E_i'+1}, \nabla^{E_i'+1})
\]
Using the Claim,

\[ \alpha(\hat{\nabla}^E) = \sum_{i=1}^{n} (-1)^{i-1} \hat{\chi}(\nabla^{E_i,\text{split}}, \hat{\nabla}^{E_i}) \]

\[ = \sum_{i=1}^{n} (-1)^{i-1}(\hat{\chi}(\nabla^{E_i,\text{split}}, \nabla^{E_i}) + \hat{\chi}(\nabla^{E_i}, \hat{\nabla}^{E_i})) \]

\[ = \sum_{i=1}^{n} (-1)^{i-1} \hat{\chi}(\nabla^{E_i}, \hat{\nabla}^{E_i}) \]

\[ = \sum_{i=1}^{n} (-1)^{i-1}(\hat{\chi}(\nabla^{E_i'}, \hat{\nabla}^{E_i'}) + \hat{\chi}(\nabla^{E_{i+1}}, \hat{\nabla}^{E_{i+1}})) = 0 \]

3. By the functoriality of \( \hat{\chi}(\nabla^E, \nabla'^E) \), this is clear.

So we have the existence. We prove the uniqueness. The proof is same as the proof of proposition 1.4.2.

Assume \( \alpha(\nabla^E) \) is any class satisfying the 3 properties. Let \( \nabla^E_t \) be any path connecting \( \nabla^{E,\text{split}} = \{ \nabla^{E_i,\text{split}} \} \) and \( \nabla^E = \{ \nabla^{E_i} \} \). Let \( \nabla^E = \nabla^E_t + dt \frac{\partial}{\partial t} \), then \( \nabla^E|_{t=0} = \nabla^{E,\text{split}}, \nabla^E|_{t=1} = \nabla^E \), so

\[ \int_{[0,1]} \chi(\nabla^E) = \int_{[0,1]} d\alpha(\nabla^E) \]

\[ = -d\int_{[0,1]} \alpha(\nabla^E) + \alpha(\nabla^E)|_{t=1} - \alpha(\nabla^E)|_{t=0} \]

\[ = -d\int_{[0,1]} \alpha(\nabla^E) + \alpha(\nabla^E) \]

Note that \( \int_{[0,1]} \chi(\nabla^E) \) is what we’ve constructed.

We can now show \( \beta = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \varphi \text{Tr}_s(\frac{V}{2\sqrt{t}} \exp(-A^2))dt \) is a Chern-Simon class.

**Theorem 1.4.4.** We have an exact sequence of vector bundles

\[ 0 \rightarrow \ker V_+ \rightarrow E_+ \xrightarrow{V_+} E_- \rightarrow \ker V_- \cong \ker V_+ \rightarrow 0 \]

and corresponding connections \( \nabla^{\ker V_+}, \nabla^{E_+}, \nabla^E, \nabla^{\ker V_-} \).

\( \beta = -\hat{\chi} \) in \( \Omega^\text{odd}/d\Omega^\text{even} \).

**Proof.** By proposition 1.4.2, \( \beta \) is a real odd form, and

\[ d\beta = \chi(\nabla^E) - \chi(\nabla^{\ker V_+}) = -(\chi(\nabla^{\ker V_+}) - \chi(\nabla^{E_+}) + \chi(\nabla^{E_-}) - \chi(\nabla^{\ker V_-})) \]

The functorial property is easy. We need to prove \( \beta \) satisfies the 2nd property. If the sequence splits, we assume:

\( \nabla^{E_+} = \nabla^{\ker V_+} \oplus \nabla^{(\ker V_+)^\perp}, \nabla^{E_-} = \nabla^{\ker V_-} \oplus \nabla^{(\ker V_-)^\perp}, \nabla^{(\ker V_+)^\perp} = V_+^\perp \nabla^{(\ker V_-)^\perp} \)

Under the splitting \( E = E_+ \oplus E_- = \ker V_+ \oplus (\ker V_+)^\perp \oplus \ker V_- \oplus (\ker V_-)^\perp \), we have

\[ \nabla^E = \begin{pmatrix} \nabla^{\ker V_+} & 0 & 0 \\ 0 & \nabla^{(\ker V_+)^\perp} & 0 \\ 0 & 0 & \nabla^{\ker V_-} \end{pmatrix} \]
\[ V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & V_- \\ 0 & V_+ & 0 \end{pmatrix}, \quad \nabla^E V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \nabla^E V_- \\ 0 & \nabla^E V_+ & 0 \end{pmatrix} = 0 \]

\[ A_t^2 = \nabla^{E, 2} + \sqrt{t} \nabla^E V + tV^2 \]

\[ = \begin{pmatrix} \nabla^{ker V_+}, 2 & 0 & 0 \\ 0 & \nabla^{(ker V_-)^+, 2 + tV_-V_+} & 0 \\ 0 & 0 & \nabla^{(ker V_-^+), 2 + tV_+V_-} \end{pmatrix} \]

So \( Tr_s(\exp(-A_t^2)) = Tr_s \left( \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \right) = 0 \), so \( \beta = 0 \).

So \( \beta \) is indeed the Chern-Simon class of the sequence. \( \square \)
Chapter 2
Atiyah-Singer Family Index Theorem

2.1 Family Index Theorem

\[ X \rightarrow M \rightarrow S \] is a fibration with compact fibre \( X \) of \( \text{dim} \ 2l \). \( TX \) (subbundle of \( TM \)) is even dimensional, orientable and spin. This is a global condition over \( M \). \( g^{TX} \) is a metric on \( TX \). \( E \) is a complex vector bundle on \( M \) with Hermitian metric \( g^{E} \) and unitary connection \( \nabla^{E} \).

\( S^{TX} = S^{TX}_{+} \oplus S^{TX}_{-} \) is spinor bundle of \( TX \), \( \text{rank}(S^{TX}) = 2^{\frac{2l}{2}} = 2^{l} \). \( \forall s \in S \), \( D^{X}_{s} \) is the Dirac operator along the fibre \( X_{s} \) with twisting bundle \( E|_{X_{s}} \), which acts on \( \Gamma(X_{s}, S^{TX} \otimes E|_{X_{s}}) \).

\[ D^{X}_{s} = \begin{pmatrix} 0 & D^{X}_{+,s} \\ D^{X}_{-,s} & 0 \end{pmatrix} \] is self-adjoint, it’s Fredholm. \( \text{Ind}(D^{X}_{+,s}) = \text{dim ker } D^{X}_{+,s} - \text{dim ker } D^{X}_{-,s} \in \mathbb{Z} \) does not depend on \( s \) by the homotopy invariance of the index. By Atiyah-Singer index theorem,

\[ \text{Ind}(D^{X}_{+,s}) = \int_{X_{s}} \hat{A}(TX) \text{ch}(E) \]

The idea of family index is to see \( \text{Ind}(D^{X}_{+,s}) \) as “vector bundle” on \( S \), s.t. “rank”(\( \text{Ind}(D^{X}_{+,s}) \))=classical index.

Suppose that \( \text{ker } D^{X}_{+,s} \) and \( \text{ker } D^{X}_{-,s} \) have constant dimensions, then one can prove easily that \( \text{ker } D^{X}_{+,s} \) are the fibres of smooth vector bundles. Let \( \mathcal{V}(M) \) denote the class of vector bundles on \( M \), then we have

\[ \mathcal{V}(M) \rightarrow \mathcal{V}(S) \]
\[ E \rightarrow \text{ker } D^{X}_{+,s} \oplus \text{ker } D^{X}_{-,s} \]

One of the purposes of topological K-theory is to make sense such a \( \oplus \) sign, the idea is analogous to the idea of extending natural numbers \( \mathbb{N} \) to integers \( \mathbb{Z} \). But for vector bundles we must consider stable equivalent relations rather than equality.

**Definition 2.1.1.** \( E, E' \in \mathcal{V}(M), E \sim E' \Leftrightarrow \exists F \in \mathcal{V}(M), s.t. E \oplus F \cong E' \oplus F \). \( E \) and \( E' \) are called stable equivalent. Let \( \mathcal{W}(M) = \mathcal{V}(M)/\sim \). Define an equivalent relation on \( \mathcal{V}(M) \times \mathcal{V}(M) \):

\( (E, E') \sim (F, F') \Leftrightarrow E \oplus E' = F \oplus F' \in \mathcal{W}(M) \Leftrightarrow \exists H \in \mathcal{V}(M), s.t. E \oplus F' \oplus H = E' \oplus F \oplus H \)

The \( K \)-group of \( M \) is defined to be \( K(M) = \{(E, E') : E, E' \in \mathcal{V}(M)\}/\sim \).

**Remark 6.** There are several equivalent definitions of \( K(M) \). See ref.

If \( E \sim E' \), then \( ch(E) = ch(E') \). It’s clear the map \( ch : \mathcal{V}(M) \rightarrow H^{*}(M) \) induces

\[ ch : K(M) \rightarrow H^{*}(M) \]
Assume again that \( \ker D^X_+ \) are smooth vector bundles, (it’s sufficient to assume constant dimensions) then

\[
\text{Ind}(D^X_+) = \ker D^X_+ - \ker D^X_+ \in K(S)
\]

If the assumption is not verified, we can add a finite dimensional space, and find a surjective homomorphism \( \tilde{D}_+: S^TX_+ \oplus \mathbb{C}^n \to S^TX_+ \), for which the assumption is verified, then

\[
\text{Ind}(D^X_+) = \text{Ind}(\tilde{D}_+) - \mathbb{C}^n \in K(S)
\]

**Theorem 2.1.1** (Family Index Theorem).

\[
\text{ch(Ind}(D^X_+)) = \pi_*[\hat{A}(TX)\text{ch}(E)]
\]

in \( H^{even}(S, \mathbb{Q}) \), i.e. the diagram commutes:

\[
\begin{array}{ccc}
K(M) & \xrightarrow{\text{Ind}(D^X_+)/ \pi_*} & K(S) \\
ch & \downarrow & ch \\
H^{even}(M, \mathbb{Q}) & \xrightarrow{\pi_*[\hat{A}(TX)]} & H^{even}(S, \mathbb{Q})
\end{array}
\]

**Example 2.1.1.** If \( S \) is a single point, \( \mathcal{V}(S) = \) vector spaces, \( K(S) = \mathbb{Z}, \text{ch}(E \oplus F) = \dim E - \dim F \).

We get the index theorem

\[
\text{Ind}(D^X_+) = \int_X \hat{A}(TX)\text{ch}(E)
\]

### 2.2 Consequences of the Family Index Theorem

Assume \( S \) is even dimensional, compact and spin and we have a given horizontal space \( T^H M \) which is a subbundle of \( TM \). So \( T^H M = \pi^*TS, TM = T^H M \oplus TX \cong \pi^*TS \oplus TX \). Let \( g^{TM} = g^{TX} \oplus \pi^*g^{TS} \). Then \( TM \) is spin, and \( STM = S^TX \oplus \pi^*S^TS \). Since \( \hat{A} \) is multiplicative, we have \( \hat{A}(TM) = \hat{A}(TX)\pi^*\hat{A}(TS) \).

\[
\text{Ind}(D^M_+,E) = \int_M \hat{A}(TM)\text{ch}(E) = \int_M \pi^*(\hat{A}(TS))\hat{A}(TX)\text{ch}(E) = \int_S \hat{A}(TS)\pi_*[\hat{A}(TX)\text{ch}(E)] F_{L,T} = \int_S \hat{A}(TS)\text{ch}(\text{Ind} D^X_+) = \text{Ind}(D^S_+,\text{Ind} D^X_+) \tag{2.1}
\]

**Claim 2.** (2.1) is equivalent to family index theorem:

**Proof.** Let \( F \) be a vector bundle on \( S \), (2.1) is valid when \( E \leadsto E \oplus \pi^*F \). By the same computation of (2.1)

\[
\text{Ind}(D^M_+,E \oplus \pi^*F) = \int_S \hat{A}(TS)\text{ch}(F)\pi_*[\hat{A}(TX)\text{ch}(E)]
\]

Note that \( \text{Ind}(D^X_+,E \oplus \pi^*F) = F \oplus \text{Ind}(D^X_+,E) \), so by A-S index theorem,

\[
\text{Ind} D^S_+,\text{Ind}(D^X_+,E \oplus \pi^*F) = \int_S \hat{A}(TS)\text{ch}(F)\text{ch}(\text{Ind} D^X_+) \tag{2.1}
\]

Since \( \text{ch}(F) \) generate the full \( H^{even}(S, \mathbb{Q}) \) as \( F \) varies, compare the above two equation, we have

\[
\hat{A}(TS)\pi_*[\hat{A}(TX)\text{ch}(E)] = \hat{A}(TS)\text{ch}(\text{Ind} D^X_+)
\]

Since \( \hat{A} = 1 + \cdots \) is invertible in \( H^{even}(S, \mathbb{Q}) \), so

\[
\pi_*[\hat{A}(TX)\text{ch}(E)] = \text{ch}(\text{Ind} D^X_+)
\]

\[\square\]

**Remark 7.** Atiyah-Singer proved the family index theorem by proving (2.1).
2.3 Adiabatic Limit

From now on, we assume \( g_\varepsilon = g_{T^M}^\varepsilon = g^{TX} \oplus \frac{1}{\varepsilon} \pi^* g^{TS} \), then \( T^H M = (TX) \perp \pi^* TS \), and \( \pi : M \to S \) is a Riemannian submersion. Let \( \nabla^{TM,\varepsilon} \) denote the Levi-Civita connection of \( g_{T^M}^\varepsilon \), \( \nabla^{TM,L} = \nabla^{TM,1} \) denote the Levi-Civita connection of \( g^{TM} = g^{TX} \oplus \pi^* g^{TS} = g_{T^M}^1 \).

We want to calculate the Dirac operator \( D_{T^M}^\varepsilon \) using the metric \( g_{T^M}^\varepsilon \).

**Theorem 2.3.1.** As \( \varepsilon \to 0 \), \( \nabla^{TM,\varepsilon} = \left( \begin{array}{c} \nabla^{TX} \\ \pi^* \nabla^{TS} \end{array} \right) + O(\varepsilon) =: \nabla^{TM,0} + O(\varepsilon) \).

**Proof.** Let \( \forall U \in TS, U^H \) denotes the lift of \( U \) to \( T^H M \). If \( U \) is a smooth vector field on \( S \), then \( U^H \) is a smooth vector field on \( M \). Let \( \varphi_t, \psi_t \) be one-parameter transformation group generated by \( U \) and \( U^H \), then \( \varphi_t \circ \pi = \pi \circ \psi_t \), and \( \psi_t : X_s \to X_{\varphi_t(s)} \) is a diffeomorphism. So if \( V \) is a smooth section of \( TX \), then \( [U^H, V] \in TX \).

By properties of connection, we can assume \( X \) is either vertical or \( U = U^H \in T^H M \) in the following proof. We have defining equation of the Levi-Civita connection:

\[
2\langle \nabla_X Y, Z \rangle = X \langle Y, Z \rangle + Y \langle Z, X \rangle - Z \langle X, Y \rangle + \langle Z, [X, Y] \rangle - \langle Y, [X, Z] \rangle - \langle X, [Y, Z] \rangle \tag{2.2}
\]

Using this formula, we calculate all possible cases:

1. \( Y, Z \) are vertical, \( X \) is vertical

\[
\langle \nabla_X^{TM,\varepsilon} Y, Z \rangle_{g^{TX}} = \langle \nabla_X^{TX} Y, Z \rangle_{g^{TX}}
\]

2. \( Y, Z \) are vertical, \( U = U^H \) is horizontal

\[
2\langle \nabla_{U^H}^{TM,\varepsilon} Y, Z \rangle_{g^{TX}} = U^H \langle Y, Z \rangle_{g^{TX}} + \langle Z, [U^H, Y] \rangle_{g^{TX}} - \langle Y, [U^H, Z] \rangle_{g^{TX}}
\]

\[
= 2\langle [U^H, Y], Z \rangle + (L_{U^H} g^{TX})(Y, Z)
\]

\[
= 2\langle [U^H, Y], Z \rangle + \langle (g^{TX})^{-1} L_{U^H} g^{TX}(Y), Z \rangle
\]

So \( P^{TX} \nabla_{U^H}^{TM,\varepsilon} Y = [U^H, Y] + \frac{1}{2}(g^{TX})^{-1} L_{U^H} g^{TX} Y \).

3. \( Y = V^H \) is horizontal, \( Z \) is vertical, \( X \) is vertical

\[
2\langle \nabla_X^{TM,\varepsilon} V^H, Z \rangle_{g^{TX}} = V^H \langle X, Z \rangle_{g^{TX}} - \langle V^H, [X, Z] \rangle_{g^{TX}} - \langle X, [V^H, Z] \rangle_{g^{TX}} = (L_{V^H} g^{TX})(X, Z)
\]

So \( P^{TX} \nabla_X^{TM,\varepsilon} V^H = \frac{1}{2}(g^{TX})^{-1} (L_{V^H} g^{TX})(X) \).

4. \( Y = V^H \) is horizontal, \( Z \) is vertical, \( U = U^H \) is horizontal

\[
2\langle \nabla_{U^H}^{TM,\varepsilon} V^H, Z \rangle_{g^{TX}} = \langle P^{TX}[U^H, V^H], Z \rangle_{g^{TX}}
\]

So \( P^{TX} \nabla_{U^H} V^H = \frac{1}{2} P^{TX}[U^H, V^H] \).

5. \( Y \) is vertical, \( Z = W^H \) is horizontal, \( X \) is vertical

\[
2\langle \nabla_X^{TM,\varepsilon} Y, W^H \rangle_{g^{TX}} = -\varepsilon (L_{W^H} g^{TX})(X, Y)
\]

So \( P^{TM,M} \nabla_X^{TM,\varepsilon} Y = O(\varepsilon) \).

6. \( Y \) is vertical, \( Z = W^H \) is horizontal, \( U = U^H \) is horizontal

\[
2\langle \nabla_{U^H}^{TM,\varepsilon} Y, W^H \rangle_{g^{TX}} = -\varepsilon \langle Y, P^{TX}[U^H, W^H] \rangle_{g^{TX}}
\]

So \( P^{TM,M} \nabla_{U^H}^{TM,\varepsilon} Y = O(\varepsilon) \).
7. $Y = V^H$, $Z = W^H$ are horizontal, $X$ is vertical

$$2\langle \nabla^E_{X} V^H, W^H \rangle_{\pi^* g_T} = -\varepsilon\langle X, P^T X [V^H, W^H] \rangle_{g_T}$$

So $P^T H \nabla^E_{X} V^H = O(\varepsilon)$.

8. $Y = V^H$, $Z = W^H$ are horizontal, $X = U^H$ is horizontal

$$\langle \nabla^E_{U^H} V^H, W^H \rangle_{\pi^* g_T} = \langle \nabla^E_{U^H} V, W \rangle_{g_T}$$

So $P^T H \nabla^E_{U^H} V^H = (\nabla^E_{U^H} V)^H$.

Write in matrix,

$$\nabla^E_{U^H} = \begin{pmatrix} P^T X \nabla^E_{X} & P^T X & P^T X \nabla^E_{X} & P^T H M \\ P^T H M \nabla^E_{X} & P^T X & P^T H M \nabla^E_{X} & P^T H M \end{pmatrix} = \begin{pmatrix} \nabla^E_{X} & \pi^* \nabla^E_{S} + O(\varepsilon) \\ O(\varepsilon) & \pi^* \nabla^E_{S} + O(\varepsilon) \end{pmatrix}$$

where $* = P^T X \nabla^E_{X} P^T H M$ is determined by above calculations in cases 3 and 4. The theorem is proved. \qed

From above calculations in cases 1 and 2, we see that

**Theorem 2.3.2.** The connection $\nabla^E_{X} = P^T X \nabla^E_{X} P^T X$ is characterized by the following two properties:

1. $\nabla^E_{X}$ restricts to the Levi-Civita connection of $(X, g^T X)$ along the fibre $X$.

2. $\forall U \in TS, Y \in TX$, $\nabla^E_{U} Y = [U^H, Y] + \frac{1}{2}(g^T X)^{-1} L_{U^H} g^T X Y$.

**Remark.** $\nabla^E_{X}$ preserves the metric $g^T X$. In general, if $\nabla^E$ is any connection on an Hermitian vector bundle $E$, we can modify it to be metric preserving: $\nabla^E = \nabla^E + \frac{1}{2}(g^E)^{-1}(\nabla^E g^E)$.

Let $\nabla^S = \nabla^E \oplus \pi^* \nabla^E$ denote the splitting connection, we write

$$\nabla^S = \nabla^E_{X} - \nabla^S, S = \nabla^E_{X} - \nabla^S$$

**Proposition 2.3.1.** Assume $\nabla^{TM,L}$ is the Levi-Civita connection of $g^T M$, and $\nabla$ is a metric preserving connection with torsion $T$. Let $\nabla^{TM,L} = \nabla + S$, then

$$2\langle S(X) Y, Z \rangle = \langle T(X, Z), Y \rangle + \langle T(Y, Z), X \rangle - \langle T(X, Y), Z \rangle$$

We first evaluate the torsion $T$ of $\nabla^S$.

**Proposition 2.3.2.**

1. $T$ takes value in $TX$;

2. $T$ vanishes on $TX \times TX$;

3. $T(U^H, V^H) = -P^T X [U^H, V^H]$;

4. $T(U^H, A) = \frac{1}{2}(g^T X)^{-1} L_{U^H} g^T X A, U \in TS, A \in TX$.  


The proof is straightforward computation. By above two proposition, we have the properties of \( S \) and \( S^\varepsilon \):

**Proposition 2.3.3.**  
1. \( P^{TX} S^\varepsilon = P^{TX} S \)

2. \( P^{TH} M S^\varepsilon = \varepsilon P^{TH} M S \)

3. \( S(\cdot) \) maps \( TX \) into \( TH M \)

4. \( \langle S(U^H)V^H, W^H \rangle = 0 \), for \( U, V, W \in TS \)

**Proof.**  
1. by (2.3)

2. by (2.3) and properties 3,4 in proposition 2.3.2

3. because \( \nabla^{TX} = P^{TX} \nabla^{TM,L} P^{TX} = P^{TX} \oplus P^{TX} \)

4. by (2.3)

\[ \square \]

**Remark 9.** The proposition also follows from the proof of theorem (2.3.1), and the formula

\[ S^\varepsilon = \begin{pmatrix} 0 & * \\ O(\varepsilon) & O(\varepsilon) \end{pmatrix} \]

**Theorem 2.3.3.**  
\[ D^M_\varepsilon = D^X + \sqrt{\varepsilon} D^H - \frac{\varepsilon c(T^H)}{4} \quad (2.4) \]

- \( D^X \) is the Dirac operator along the fibre: \( D^X = \sum_i c(e_i) \nabla^{^qTX \otimes E}_{e_i}, \{ e_i \} \) is an orthonormal basis of \( TX \).

- \( T^H \) is the HH component of \( T \): \( T^H(U^H, V^H) = -P^{TX}[U^H, V^H], \) and

\[ c(T^H) = \frac{1}{2}(T^H(f^H_\alpha, f^H_\beta), e_i) c(f_\alpha) c(f_\beta) c(e_i) \]

\( \{ f_\alpha \} \) is an orthonormal basis of \( TS \).

- \( D^H = c(f_\alpha)(\nabla^{^qTX \otimes ^qTS \otimes E}_{f^H_\alpha} + k(f_\alpha)), k(U) \triangleq \frac{1}{2} \text{div}^X(U^H) = \frac{1}{4} Tr[(g^{TX})^{-1} L_{\xi^H} g^{TX}] \)

**Proof.**  
\[ D^M_\varepsilon = c(e_i) \nabla^\varepsilon_{e_i} + c(f_\alpha) \nabla^\varepsilon_{\sqrt{\pi} f^H_\alpha} \quad (2.5) \]

\( \nabla^\varepsilon \) is the induced connection on the spinor bundle \( S^T \) = \( S^{TX} \otimes \pi^{*} ST^S \) by the Levi-Civita connection \( \nabla^{TM,\varepsilon} \). Note that the Clifford multiplication doesn’t change when rescaling the horizontal metric. We have \( \nabla^{TM,\varepsilon} = \nabla^{TM} + S^\varepsilon \), and Lie algebra homomorphism:

\[ \text{so}(n) \quad \rightarrow \quad \text{spin}(n) \subset \text{Cl}_n \]

\[ A \quad \rightarrow \quad c(A) = \frac{1}{4} \langle A e_i, e_j \rangle c(e_i) c(e_j) \]

so we get

\[ \nabla^\varepsilon = \nabla^{STX} + \nabla^{STS} + \frac{1}{4}(S^\varepsilon (\cdot) e_i, e_j) g_\varepsilon(c(e_i) c(e_j)) \frac{1}{4}(S^\varepsilon (\cdot) \sqrt{\varepsilon} f^H_\alpha, \sqrt{\varepsilon} f^H_\beta) g_{\alpha} + \frac{1}{2}(S^\varepsilon (\cdot) \sqrt{\varepsilon} f^H_\alpha, e_i) g_c(f_\alpha) c(e_i) \]

\[ (2.6) \]

\( \nabla^{STX} \) and \( \nabla^{STS} \) is the connection on spinors induced by \( \nabla^{TX} \) and \( \nabla^{TS} \). By (2.3), proposition 2.3.3, and proposition 2.3.2
Lemma 2. Let $\langle s^\varepsilon(e_i, e_j)_{g_{\varepsilon}} = 0$

- $(S^\varepsilon(\cdot)\sqrt{f_{\alpha}^H \cdot \sqrt{f_{\beta}^H}})_{g_{\varepsilon}} = \varepsilon(S(\cdot) f_{\alpha}^H, f_{\beta}^H)$

$c(e_i) \cdot \frac{1}{4} (S(e_i) f_{\alpha}^H, f_{\beta}^H) c(f_{\alpha}) c(f_{\beta}) c(e_i) = \frac{1}{8} (T(f_{\alpha}^H, f_{\beta}^H), e_i) c(f_{\alpha}) c(f_{\beta}) c(e_i)$

$(S(f_{\alpha}^H) f_{\beta}^H)_{f_{\alpha}^H, f_{\beta}^H} = 0$.

- $(S^\varepsilon(\cdot)\sqrt{f_{\alpha}^H \cdot \sqrt{f_{\beta}^H}} e_i)_{g_{\varepsilon}} = \varepsilon(S(\cdot) f_{\alpha}^H, e_i)$

$c(f_{\alpha}) \cdot \frac{1}{4} (S(f_{\alpha}^H) f_{\beta}^H, e_i) c(f_{\beta}) c(e_i) = -\frac{1}{4} (T(f_{\alpha}^H, f_{\beta}^H), e_i) c(f_{\alpha}) c(f_{\beta}) c(e_i)$

$c(e_i) \cdot \frac{1}{2} (S(e_i) f_{\alpha}^H, e_j) c(f_{\alpha}) c(e_j) = \frac{1}{4} ((T(f_{\alpha}^H, e_j), e_i) - (T(e_i, f_{\alpha}^H), e_j)) c(e_i) c(f_{\alpha}) c(e_j)$

$= \sum_i \frac{1}{2} (T(f_{\alpha}^H, e_i), e_i) c(f_{\alpha}) + \sum_i \cdots$

$= \frac{1}{2} (T(f_{\alpha}^H, e_i), e_i) c(f_{\alpha})$

$(T(f_{\alpha}^H, e_i), e_i) = (\frac{1}{2} (g^{TX})^{-1} (L_{f_{\alpha}^H} g^{TX} e_i, e_i) = \frac{1}{2} Tr[(g^{TX})^{-1} L_{f_{\alpha}^H} g^{TX}])$

Lemma 2. Let $dV_X$ be the volume element along the fibre, then

$$\text{div}^X (U^H) = \frac{L_{U^H} dV_X}{dV_X} = \frac{1}{2} Tr[(g^{TX})^{-1} L_{U^H} g^{TX}]$$

proof of the lemma. $\{e_i\}$ is an orthonormal basis of $TX$, $\{e^i\}$ is the dual basis. We calculate

$$L_{U^H} dV_X = L_{U^H} (e^1 \wedge \cdots \wedge e^n) = \sum_i e^1 \wedge \cdots \wedge L_{U^H} e^i \wedge \cdots \wedge e^n$$

$$= \sum_i e^1 \wedge \cdots \wedge (\sum_j \langle L_{U^H} e_j, e_i \rangle e^j) \wedge \cdots \wedge e^n$$

$$= -(\sum_i \langle L_{U^H} e_i, e_i \rangle) dV_X$$

$$Tr[(g^{TX})^{-1} L_{U^H} g^{TX}] = Tr g^{TX} (L_{U^H} g^{TX}) = \sum_i (L_{U^H} g^{TX})(e_i, e_i)$$

$$= \sum_i (U^H(e_i, e_i) - \langle L_{U^H} e_i, e_i \rangle - \langle e_i, L_{U^H} e_i \rangle)$$

$$= -2 \sum_i \langle L_{U^H} e_i, e_i \rangle$$

Put the results together, finally we have

$$D^{M}_{\varepsilon} = c(e_i) \nabla^{TX}_{e_i} + \sqrt{\varepsilon} c(f_{\alpha})(\nabla^{S_{TX} \otimes E}_{f_{\beta}} + k(U)) - \frac{\varepsilon}{8} (T(f_{\alpha}^H, f_{\beta}^H), e_i) c(f_{\alpha}) c(f_{\beta}) c(e_i)$$

$$= D^X + \sqrt{\varepsilon} D^H - \frac{\varepsilon c(T^H)}{4}$$

2.4 Levi-Civita Superconnection

Definition 2.4.1. $\forall s \in S$, $H_s = \Gamma(X_s, (S^{TX} \otimes E)|_{X_s})$. $H_s = H_{s,+} \oplus H_{s,-}$.

$H$ is an infinite dimensional $Z_2$-graded vector bundle with $L^2$ Hermitian metric.
Remark 10. Formally, this is obtained by changing Clifford variable \( c(f_\alpha) \) in \( D^M \) to Grassman variable \( f^\alpha \). In fact, it's obtained by Getzler rescaling: in the expression of \( D^M_\varepsilon \), we substitute:

\[
c(f_\alpha) \rightarrow \frac{f^\alpha \wedge}{\varepsilon} - \varepsilon f_\alpha
\]

and let \( \varepsilon \rightarrow 0 \), then we get the Levi-Civita superconnection \( A \).

We want to calculate the curvature of \( A \). We use the Lichnerowicz formula for \( D^M_\varepsilon \):

\[
D^{M,2}_\varepsilon = -\Delta^M_\varepsilon + \frac{K^M_\varepsilon}{4} + R^E
\]

(2.7)

where \( \Delta^M_\varepsilon = (\nabla^\varepsilon_{e_i})^2 + \varepsilon (\nabla^\varepsilon_{f_\alpha})^2 \) is the Bochner Laplacian, \( \nabla^\varepsilon \) is given by equation (2.6). \( K^M_\varepsilon \) is the scalar curvature of \( g^T_M \), and

\[
R^E = \frac{1}{2}c(e_i)c(e_j)R^E(e_i,e_j) + \frac{\varepsilon}{2}c(e_i)c(f_\alpha)c(f_\beta)R^E(f_\alpha,f_\beta)
\]

The idea is to change Clifford variable \( c(f_\alpha) \) to \( f^\alpha \), note that

- \((\sqrt{\varepsilon}c(f_\alpha))^2 = -\varepsilon \rightarrow (f^\alpha)^2 = f^\alpha \wedge f^\alpha = 0\)

\[
\]
• $\alpha \neq \beta$, $\sqrt{\varepsilon}c(f_\alpha)\sqrt{\varepsilon}c(f_\beta) = \varepsilon c(f_\alpha)c(f_\beta) \to f^\alpha \wedge f^\beta$

So if the power of $\sqrt{\varepsilon} > (=)$ the length of Clifford variables, then the term is killed(survives).

$\nabla^T M, \varepsilon = \nabla^T M + S^\varepsilon, \nabla^T M, \varepsilon, 2 = \nabla^T M, 2 + \nabla^T M S^\varepsilon + [S^\varepsilon, S^\varepsilon]$.

**Definition 2.4.4.**

$$\alpha_t = \varphi Tr_s(\exp(-A^2_t))$$

**Theorem 2.4.1.**

1. The $\alpha_t$ are real, even, closed forms on $S$.

   $$[\alpha_t] = ch(Ind(D^X_+))$$

2. As $t \to 0$, $\alpha_t = \pi_* [\hat{A}(\nabla^TX)ch(\nabla^E)] + O(\sqrt{t})$

3. If $dim ker D^X_\pm$ is locally constant, as $t \to \infty$,

   $$\alpha_t = ch(\nabla^{ker D^X}) + O\left(\frac{1}{\sqrt{t}}\right)$$

   $$\nabla^{ker D^X} = P^{ker D^X} \nabla^H.$$

**Remark 11.**

1. The theorem implies the family index theorem:

   $$ch(Ind(D^X_+)) = [\alpha_t] = \pi_* [\hat{A}(TX)ch(E)]$$

2. The theorem asserts the following diagram commutes.
Chapter 3
Determinant Bundle

3.1 Finite Dimensional Case

Let \( \mathcal{C} \) denote the category of complex line (complex vector space of dim 1) up to canonical isomorphism. For example

- \( \mathbb{C} \) denotes the canonical line. If \( \lambda \) has a canonical nonzero element \( s \in \lambda \), then \( \lambda \) can be canonically identified with \( \mathbb{C} \): \( a \in \mathbb{C} \mapsto as \in \lambda \).

- If \( \lambda, \mu \) are complex lines, then canonically \( \lambda \otimes \mu \sim \mu \otimes \lambda \), \( a \otimes b \mapsto b \otimes a \). The operator \( \otimes \) is commutative, associative and has a neutral element:
  - \( \lambda \otimes \mathbb{C} \sim \lambda \); \( s \otimes 1 \mapsto s \).
  - \( \lambda^* \otimes \lambda \sim \mathbb{C} \); choose any \( s \neq 0 \in \lambda \), there is \( s^{-1} \otimes s \) such that \( \langle s^{-1}, s \rangle = 1 \). \( s^{-1} \otimes s \) does not depend on \( s \), so it’s a canonical nonzero element in \( \lambda^* \otimes \lambda \). We define \( \lambda^{-1} = \lambda^* \).

**Definition 3.1.1.** \( E \) is a finite dimensional complex vector space

\[
\det E \overset{\Delta}{=} \wedge^\text{max} E
\]

\( \wedge^\text{max} E \) means the elements of max degree in \( \wedge^* E \).

**Remark 12.** More correctly, \( \mathcal{C} = \{ (\lambda, \pm) \}, \mathcal{V} = \{ \text{finite dimensional complex vector space} \} \).

\[
\hat{\det} : \mathcal{V} \longrightarrow \mathcal{C}, \quad \hat{\det} E = (\det E, (-1)^{\text{dim} E})
\]

then \( \hat{\det}(E \oplus F) = \hat{\det} E \hat{\otimes} \hat{\det} F \), where \( \hat{\otimes} \mu \equiv \mu \hat{\otimes} \lambda \) is given by \( a \hat{\otimes} b \mapsto (-1)^{\text{deg} a \cdot \text{deg} b} b \hat{\otimes} a \).

Let \( E : 0 \rightarrow E_0 \rightarrow E_1 \rightarrow 0 \), \( \lambda = \det E \overset{\Delta}{=} \det (E_0)^{-1} \otimes \det E_1 \). \( V \) induces \( \det V : \det E_0 \rightarrow \det E_1 \), so \( \det V \in \lambda \).

Let \( E \) be a \( \mathbb{Z}_2 \)-graded vector bundle with Hermitian metric \( g^E = g^{E^+} \oplus g^{E^-} \), unitary connection \( \nabla^E = \nabla^{E^+} \oplus \nabla^{E^-} \). We have the induced metric \( g^\lambda \) and unitary connection \( \nabla^\lambda \). Assume \( V = \begin{pmatrix} 0 & V_- \\ V_+ & 0 \end{pmatrix} \) is self-adjoint and \( V_+ \) is invertible. Let \( \{ e_i \} \) be an orthonormal basis of \( E_+ \)

\[
(\nabla^\lambda_X (\det V_+))(e_1 \wedge \cdots \wedge e_n) = \nabla^X_X (V_+(e_1) \wedge \cdots \wedge (e_n)) - \det V_+ (\sum_i e_1 \wedge \cdots \wedge \nabla^E_X e_i \wedge \cdots \wedge e_n)
\]

\[
= \sum_i V_+(e_1) \wedge \cdots \wedge (\nabla^E_X (V_+(e_i))) - V_+(\nabla^E_X e_i)) \wedge \cdots \wedge V_+(e_n)
\]

\[
= \sum_i V_+(e_1) \wedge \cdots \wedge V_+(V_+^{-1} \nabla^X_X V_+)(e_i) \wedge \cdots \wedge V_+(e_n)
\]

\[
= \text{Tr}(V_+^{-1} \nabla^X_X V_+) \cdot \det V_+ (e_1 \wedge \cdots \wedge e_n)
\]
So in anticommutative sign rule
\[ \nabla^{A} (\det V) = -(\det V) \cdot \text{Tr}(V^{-1} \nabla E V) \]  \hfill (3.1)

Note that
\[ V^{-1} \nabla E V = \begin{pmatrix} 0 & V^{-1} \nabla E V_- \\ V^{-1} & 0 \end{pmatrix} = \begin{pmatrix} V^{-1} \nabla E V_+ & 0 \\ 0 & V^{-1} \nabla E V_- \end{pmatrix} \]

So
\[ \text{Tr}(V^{-1} \nabla E V) = \frac{1}{2} \text{Tr}(V^{-1} \nabla E V) + \frac{1}{2} \text{Tr}_{s}(V^{-1} \nabla E V) \]  \hfill (3.2)

Since \((V^{-1} \nabla E V)^* = -(V^{-1} \nabla E V)^{-1} + \frac{1}{2} \text{Tr}(V^{-1} \nabla E V)\) is real, \(\frac{1}{2} \text{Tr}_{s}(V^{-1} \nabla E V)\) is purely imaginary. Moreover, \(\text{Tr}(V^{-1} \nabla E V)\) is exact: \(\det V \in (\det E)^{*} \otimes \det E \cong \mathbb{C}\) is a well defined function, so it’s easy to see
\[ \text{Tr}(V^{-1} \nabla E V) = \frac{d(\det V)}{\det V} = d \log |\det V| = 2d \log |\det V_{+}| \]  \hfill (3.3)

Let \(r^{A}\) be the curvature of \(\nabla^{A}\), by formula (3.1),(3.2),(3.3)
\[ r^{A} = \frac{d \nabla^{A} \det V}{\det V} = -d \text{Tr}(V^{-1} \nabla E V) = -\frac{1}{2} \text{Tr}_{s}(V^{-1} \nabla E V) \]  \hfill (3.4)

**Lemma 3.** \(c_{1}(\det E) = \text{ch}(E)^{(2)} = c_{1}(E)\)

in definition 1.4.1, we defined
\[ \beta = \frac{1}{\sqrt{2\pi i}} \int_{0}^{\infty} \varphi \text{Tr}_{s}(V \frac{1}{2\sqrt{t}} \exp(-A^{2}_{t})) dt \]

where \(A^{2}_{t} = \nabla^{E,2} + \sqrt{t}\nabla E V + tV^{2}\).

Note that \((\frac{(-1)^{n}}{n!}) \text{Tr}_{s}(V(\sqrt{t}\nabla^{2} V + tV^{2})^{n})^{(1)} = \frac{(-1)^{n}}{(2\pi i)^{n}} \text{Tr}_{s}(V \nabla E V (tV^{2})^{n-1})\),
\[ \text{Tr}_{s}(V \frac{1}{2\sqrt{t}} \exp(-A^{2}_{t}))^{(1)} = -\frac{1}{2} \text{Tr}_{s}(V \nabla E V) \exp(-tV^{2}) \]

\[ \beta^{(1)} = -\frac{1}{2\pi i} \cdot \frac{1}{2} \int_{0}^{\infty} \text{Tr}_{s}(V \nabla E V \exp(-tV^{2})) dt - \frac{\sqrt{-1}}{2\pi} \cdot \frac{1}{2} \text{Tr}_{s}(V^{-1} \nabla E V) \]

By formula (3.4)
\[ d\beta^{(1)} = -\frac{\sqrt{-1}}{2\pi} r^{A} = -c_{1}(\det E) = -\text{ch}(E)^{(2)} \]  \hfill (3.5)

### 3.2 Determinant Bundle

We want to construct the determinant bundle \(\lambda\) of \(\{H = H_{+} \oplus H_{-}, D^{X}_{\lambda}\}\). If \(\ker D^{X}_{\lambda}\) has constant dimension, \(\lambda \triangleq (\det \ker D^{X}_{\lambda})^{-1} \otimes (\det \ker D^{X}_{\lambda})\). In general cases, \(\forall a > 0\),
\[ U_{a} \triangleq \{ s \in S | a \not\in Sp(D^{X,2}) \} \]

\(U_{a}\) is an open set on \(S\). \(\bigcup_{a > 0} U_{a}\) is an open covering of \(S\).

**Definition 3.2.1.** On \(U_{a}\), \(H^{\leq a} = \bigoplus_{\lambda \in Sp(D^{X,2})}^{\lambda \leq a} (\text{bundle of eigenvalue } \lambda \text{ of } D^{X,2})\).

We use other similar notations, like \(H^{(0,a)}, D^{(0,a)} = D^{X}|_{H^{(0,a)}}\), e.t..

**Proposition 3.2.1.** \(H^{\leq a}\) is a finite dimensional smooth vector bundle on \(U_{a}\). \(H^{\leq a} = H^{< a} \oplus H^{\leq a}\).
Definition 3.2.2. On $U_a$, $\lambda^{<a} = detH^{<a} = (detH_{+}^{<a})^{-1} \otimes detH_{-}^{<a}$. $\lambda^{<a}$ is a smooth line bundle on $U_a$.

The idea is that we can’t define $detH = (detH_{+})^{-1} \otimes detH_{-}$ directly, but we can patch $\lambda^{<a}$ together to get a well defined line bundle.

**Proposition 3.2.2.** $\forall s \in U_a$, $\lambda^{<a}_s \cong (det \ker D^X_s) - 1 \otimes (det \ker D^X_s)^c =: det \ker D^X_s$. This is a canonical isomorphism.

**Proof.** We have an exact sequence:

$$0 \rightarrow kerD^X_+ \rightarrow H^<a_+ \rightarrow D_+ \rightarrow H^<a_- \rightarrow kerD^X_- \rightarrow 0$$

Let $\sigma_+ \neq 0 \in det \ker D^X_+$, $\tau \neq 0 \in det H^{(0,a)}_+$, define

$$det \ker D^X \rightarrow det H^{<a}$$

$$\sigma_+^{-1} \otimes \sigma_- \rightarrow (\sigma_+ \wedge \tau)^{-1} \otimes (det D^X \tau \wedge \sigma_-)$$

This isomorphism is independent of $\tau$. In fact, $det D^{(0,a)}_+ = \tau^{-1} \otimes (det D^X \tau)$ is a nonzero canonical section of $det H^{(0,a)}_+$. So $\sigma_+^{-1} \otimes \sigma_- \rightarrow (\sigma_+^{-1} \otimes \sigma_-) \cdot (det D^{(0,a)}_+)$ is a canonical isomorphism from $det D^X$ to $det H^{<a} = (det D^X) \otimes (det H^{(0,a)}_+)$. 

$$\forall 0 < a < b, a, b \notin Sp(D^{X,2}), H^{<b} = H^{<a} \oplus H^{(a,b)}, det H^{<b} = det H^{<a} \otimes det H^{(a,b)}. det D^{(a,b)}_+$$

is a canonical nonzero section of $det H^{(a,b)}$, so we have a canonical isomorphism

$$\varphi^a_b : \lambda^{<a} \rightarrow \lambda^{<b}$$

$$\sigma \rightarrow \sigma \otimes (det D^{(a,b)}_+)$$

Since $D_+^{(a,c)} = D_+^{(a,b)} \oplus D_+^{(b,c)}$, $det D_+^{(a,c)} = det D_+^{(a,b)} \otimes det D_+^{(b,c)} \in det H^{(a,b)} \otimes det H^{(b,c)} = det H^{(a,c)}$.

So $\varphi^a_b = \varphi^b_b \circ \varphi^a_b : \lambda^{<a} \rightarrow \lambda^{<b}$. We can define the determinant line bundle:

**Definition 3.2.3.** The determinant line bundle $\lambda$ is the complex line bundle obtained by pasting $(\lambda^{<a}, U_a)$ together via the canonical isomorphisms $(\varphi^a_b : \lambda^{<a} \rightarrow \lambda^{<b})$

**Remark 13.** If $kerD^X_\pm$ have constant dimension, then

$$\lambda \cong det \ker D^X$$

**Proposition 3.2.3.** $c_1(\lambda) = \pi_*[\hat{A}(TX)ch(E)](2)$ in $H^2(S)$.

**Proof.** By the family index theorem, $ch(IndD^X) = \pi_*[\hat{A}(TX)ch(E)]$ in $H^2(S)$. So $c_1(\lambda) = c_1(det \ ker D^X) = c_1(det IndD^X) = ch(IndD^X)^{(2)} = \pi_*[\hat{A}(TX)ch(E)](2)$. 

We want to construct a metric on $\lambda$. First we have a induced metric $| \cdot |_{\lambda^{<a}}$ on $\lambda^{<a}$ by the metric of $H$. $\forall \sigma \in \lambda^{<a},$

$$|\varphi^a_b(\sigma)|_{\lambda^{<b}} = |\sigma \otimes (det D_+^{(a,b)})|_{\lambda^{<b}} = |\sigma|_{\lambda^{<a}} |det(D^{(a,b)}_+ D_+^{(a,b)})|^\frac{1}{2} H^{(a,b)}_+$$

We want to define $\| \cdot \|_{\lambda^{<a}} = | \cdot |_{\lambda^{<a}} |det D_+^{(a,\infty)}|$, then

$$\|\varphi^a_b(\sigma)\|_{\lambda^{<b}} = |\varphi^a_b(\sigma)|_{\lambda^{<b}} |det D_+^{(b,\infty)}| = |\sigma|_{\lambda^{<a}} |det D_+^{(a,b)}| |det D_+^{(b,\infty)}| = |\sigma|_{\lambda^{<a}} |det D_+^{(a,\infty)}| = \|\sigma\|_{\lambda^{<a}}$$
so the metric patch together. We have to make sense the meaning of $|\text{det}D_+^{(a,\infty)}|$ in this infinite dimensional case. 

$L = D^X_+D_+$ is a second order, elliptic, positive differential operator, we define

$$\zeta(s) = \text{Tr}^s(L^{-s}) = \sum_{\lambda \in \text{Sp}(D^X_+D_+)} \frac{1}{\lambda^s} \quad \text{(eigenvalue 0 excluded)} \quad (3.6)$$

We have the result of Seeley:

**Proposition 3.2.4.** $\zeta(s)$ is holomorphic in $\{ s \in \mathbb{C} | \text{Re} s > \frac{n}{2} \}$. It extends to a meromorphic function of $s \in \mathbb{C}$ with simple poles, which is holomorphic at $s = 0$.

**Proof.** Use the Mellin transform:

$$L^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1}e^{-tL}dt$$

$$\text{Tr}^s(L^{-s}) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1}\text{Tr}^s(e^{-tL})dt = \frac{1}{\Gamma(s)}(\int_0^{1} + \int_1^{\infty})$$

$f_1^{\infty} t^{s-1}\text{Tr}^s(e^{-tL})dt$ is uniformly convergent w.r.t $s$, so is holomorphic in $s \in \mathbb{C}$. When $t \to 0$, we have asymptotic expansion:

$$\text{Tr}(e^{-tL^2}) = \frac{A_{-2}}{t^2} + \cdots + A_0 + A_1 t \cdots + O(t^k)$$

So

$$\frac{1}{\Gamma(s)} \int_0^{1} t^{s-1}\text{Tr}^s(e^{-tL})dt = \frac{1}{\Gamma(s)} \int_0^{1} t^{s-1}\text{Tr}(e^{-tL})dt - \frac{1}{\Gamma(s+1)}(\text{dim ker} L)$$

is holomorphic when $\text{Re} s > \frac{n}{2}$. \hfill $\Box$

We also define

$$\zeta^a(s) = \text{Tr}(L^{-s}P^{(a,\infty)}) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1}\text{Tr}(e^{-tL}P^{(a,\infty)})dt$$

and $\zeta^{(a,b)}(s) = \text{Tr}(L^{-s}P^{(a,b)})$. Then $\forall 0 < a < b$, $\zeta^a(s) = \zeta^{(a,b)}(s) + \zeta^b(s)$. Note that

$$\frac{\partial \zeta^{(a,b)}(0)}{\partial s} = -\log \prod_{\lambda \in \text{Sp}(L)} \lambda - \log \text{det}L^{(a,b)} = -2\log |\text{det}D_+^{(a,b)}|$$

So $|\text{det}D_+^{(a,b)}| = \exp(-\frac{1}{2} \frac{\partial \zeta^{(a,b)}(0)}{\partial s}(0))$. We define

$$|\text{det}D^{(a,\infty)}_+| = \exp(-\frac{1}{2} \frac{\partial \zeta^a(0)}{\partial s}(0)), \quad |\text{det}^a D_+| = \exp(-\frac{1}{2} \frac{\partial \zeta(0)}{\partial s}(0))$$

then

$$|\text{det}D^{(a,\infty)}_+| = |\text{det}D^{(a,b)}_+||\text{det}D^{(b,\infty)}_+|, \quad |\text{det}^a D_+| = |\text{det}D^{(0,a)}_+||\text{det}D^{(a,\infty)}_+|$$

**Definition 3.2.4.** $\| \cdot \|_{\lambda < a} = \| \cdot \|_{\lambda < a}(\text{det}D^{(a,\infty)}_+)$ is $\text{det}D_+^{(a,\infty)}$ patch into a smooth metric $\| \cdot \|_{\lambda}$ on $\lambda$.

**Theorem 3.2.1.** Under the canonical identification of $\lambda$ with $\lambda < a$ over $U_a$, the metric $\| \cdot \|_{\lambda < a}$ has constant dimensions. In theorem ?? we see that $\ker D_+^X$ have constant dimensions. In theorem ?? we see that

$$d\tilde{\eta} = \pi_*[\hat{A}(\nabla^TX)\text{ch}(\nabla^E)] - \text{ch}(\nabla^\ker D_+^X)$$

so $d\tilde{\eta}^{(1)} = \pi_*[\hat{A}(\nabla^TX)\text{ch}(\nabla^E)]^{(2)} - c_1(\nabla^\ker D_+^X)$. 

27
Definition 3.2.5. \( \nabla^\lambda = \nabla^{\text{det ker}D^X} - 2\pi i\hat{\eta}^{(1)} \).

then
\[
c_1(\nabla^\lambda) = c_1(\nabla^{\text{det ker}D^X}) + d\hat{\eta}^{(1)} = \pi_*[\hat{A}(\nabla^TX)\hat{c}(\nabla^E)](2)
\]
Since \( 2\pi i\hat{\eta}^{(1)} \) is purely imaginary, \( \nabla^\lambda \) is unitary with respect to \( | \cdot |_\lambda \). Let \( g, g' \) be the metric associated with \( \| \cdot \|_\lambda, | \cdot |_\lambda \) and \( \theta \) the connection form of \( \nabla^\lambda \), then
\[
dg = d(\frac{g}{g'}) = d(\frac{g}{g'}) \cdot g' + \frac{g}{g'} d\log \frac{g}{g'} = g \cdot d\log \frac{g}{g'} + \frac{g}{g'}(g'\theta + \bar{g}\bar{\theta})
\]
So if we let
\[
\nabla^{\eta\lambda} = \nabla^\lambda + \frac{1}{2} d\log \frac{g}{g'}
\]
then \( \nabla^{\eta\lambda} \) is unitary with respect to \( \| \cdot \|_\lambda \).

In general case, we want to patch \( \nabla^{\lambda<\sigma} \) together. Let \( \nabla^H <\sigma \) be the orthogonal projection of \( \nabla^H \) to \( H^{<\sigma} \), which is unitary, then \( \lambda^{<\sigma} \) has an induced connection \( \nabla^{<\sigma} \) which is unitary w.r.t. \( | \cdot |_{\lambda^{<\sigma}} \).

\[
\nabla^{\lambda<\sigma}(\varphi_b^a(\sigma)) = \nabla^{\lambda<\sigma}(\sigma \otimes \text{det} D_+^{a,b}) = \nabla^{\lambda<\sigma} \sigma \otimes (\text{det} D_+^{a,b}) + \sigma \otimes (\nabla^{a,b}\text{det} D_+^{a,b})
\]
\[
= (\nabla^{\lambda<\sigma} \sigma + \frac{\nabla^{a,b}}{\text{det} D_+^{a,b}} \sigma) \otimes (\text{det} D_+^{a,b})
\]

From formula (3.1), we know that
\[
\frac{\nabla^{a,b}(\text{det} D_+^{a,b})}{\text{det} D_+^{a,b}} = -Tr[(D_+^{a,b})^{-1}\nabla^H(a,b) D_+^{a,b}]
\]
We want to define \( \nabla^{\lambda<\sigma} = \nabla^{\lambda<\sigma} + \varphi_b^a(\text{det} D_+^{a,b}) \), then
\[
\nabla^{\lambda<\sigma}(\varphi_b^a(\sigma)) = (\nabla^{\lambda<\sigma} + \varphi_b^a(\text{det} D_+^{a,b}) - Tr[(D_+^{a,b})^{-1}\nabla^H(a,b)] D_+^{a,b}) \sigma \otimes \text{det} D_+^{a,b}
\]
\[
= (\nabla^{\lambda<\sigma} \sigma - Tr[(D_+^{a,b})^{-1}\nabla^H(a,b)] D_+^{a,b}) \sigma \otimes \text{det} D_+^{a,b} = \varphi_b^a(\nabla^{\lambda<\sigma} \sigma)
\]
So the connection \( \{ \nabla^{<\sigma} \} \) patch together. First, we introduce

Definition 3.2.6.
\[
\gamma_t^a = \int_t^\infty \text{Tr}[e^{-sD^2} \nabla^H D \cdot D \cdot P^{(a,\infty)}] ds = \text{Tr}[e^{-tD^2} D^{-1} \nabla^H D \cdot P^{(a,\infty)}]
\]
\[
\delta_t^a = \int_t^\infty \text{Tr}_s[e^{-sD^2} \nabla^H D \cdot D \cdot P^{(a,\infty)}] ds = \text{Tr}_s[e^{-tD^2} D^{-1} \nabla^H D \cdot P^{(a,\infty)}]
\]
similarly we can define \( \gamma_t^{(a,b)} \) and \( \delta_t^{(a,b)} \).

As \( t \to 0 \), we have asymptotic expansions:
\[
\gamma_t^a = \int_t^1 (\text{Tr}[e^{-sD^2} \nabla^H D \cdot D \cdot P^{(a,\infty)}]) ds + \sum_{j=0}^0 \frac{dA_j \cdot s^{-1}}{j} ds
\]
\[
= \sum_{j=-j}^{-1} \frac{dA_j \cdot t^j}{j} + dA_0 \cdot \log t + \gamma_0^a + O(t)
\]
(3.8)
we have $\gamma^b_0 = \gamma^a + \gamma^{(a,b)}_t$, $\delta^b_0 = \delta^a_t + \delta^{(a,b)}_t$ and

\[
\gamma^{(a,b)}_0 = -Tr[D^{-1} \nabla H D \cdot P^{(a,b)}] = -Tr[(D^{(a,b)})^{-1} \nabla H^{(a,b)} D^{(a,b)}]
\]

\[
\delta^{(a,b)}_0 = Tr_s[D^{-1} \nabla H D \cdot P^{(a,b)}] = Tr_s[(D^{(a,b)})^{-1} \nabla H^{(a,b)} D^{(a,b)}]
\]

So $\nabla^{(a,b)} \left( \frac{detD^{(a,b)}}{detD^{(a,b)}} \right) = \frac{1}{2} (\gamma^{(a,b)}_0 - \gamma^{(a,b)}_0)$. If we let

\[
\nabla^{\lambda \leq a} = \nabla^{\lambda \leq a} + \frac{1}{2} \left( \gamma^{(a,b)}_0 - \delta^{(a,b)}_0 \right)
\]

then $\nabla^{\lambda \leq a}$ will patch together. But it may not be unitary. By equation (3.7), we compute $d \log \frac{\varphi}{\varphi} = d(\frac{\partial c^a}{\partial s}(0))$. Recall that

\[
\zeta^a(s) = Tr(L^{-s} P^{(a,\infty)}) = \frac{1}{2} Tr[D^{-s}P^{(a,\infty)}] = \frac{1}{2\Gamma(s)} \int_0^\infty t^{s-1} Tr[e^{-tD^2}P^{(a,\infty)}]dt
\]

so

\[
d\zeta^a(s) = -\frac{1}{\Gamma(s)} \int_0^\infty t^s Tr[e^{-tD^2} \nabla H D \cdot D \cdot P^{(a,\infty)}]dt
\]

\[
= -\frac{1}{\Gamma(s)} \int_0^1 t^s Tr[e^{-tD^2} \nabla H D \cdot D \cdot P^{(a,\infty)}] + \sum_{n=0}^0 dA_j \cdot t^{j-1})dt
\]

\[
= -\frac{1}{\Gamma(s)} \int_1^\infty t^s Tr[e^{-tD^2} \nabla H D \cdot D \cdot P^{(a,\infty)}] + \frac{1}{\Gamma(s)} \sum_{j=-\frac{1}{2}}^{j=-\frac{1}{2}} dA_j s + \frac{dA_0}{\Gamma(s+1)}
\]

Note that $\Gamma(s)$ has simple pole at $s = 0$, $Res(\Gamma(s),0) = 1$, and $(\frac{1}{\Gamma(s)})'(0) = 1$. By equation (3.8), we get

\[
d\left( \frac{\partial \zeta^a}{\partial s}(0) \right) = -\int_0^1 (Tr[e^{-tD^2} \nabla H D \cdot D \cdot P^{(a,\infty)}] + \sum_{n=0}^0 dA_j \cdot t^{j-1})dt
\]

\[
-\int_1^\infty Tr[e^{-tD^2} \nabla H D \cdot D \cdot P^{(a,\infty)}] + \sum_{j=-\frac{1}{2}}^{j=-\frac{1}{2}} dA_j t - \Gamma'(1)dA_0
\]

\[
= -\gamma^a_0 - \Gamma'(1)dA_0
\]

$\delta^a_0$ is purely imaginary, so $\nabla^{\lambda \leq a} = \nabla^{\lambda \leq a} + \frac{1}{2}(\gamma^a_0 - \delta^a_0) + \frac{1}{2} \Gamma'(1)dA_0$ is unitary and can be patched together.

We will see whether it satisfies $c_1(\nabla^\lambda) = \pi_\lambda [\hat{A}(\nabla^{TX} ch(\nabla^E))]^{(2)}$, we know that as $t \to 0$, $ch(A_t) = \phi Tr_s \exp(-A^2_t) = \pi_\lambda [\hat{A}(\nabla^{TX} ch(\nabla^E))] + O(\sqrt{t})$. So

\[
[\phi Tr_s \exp(-A^2_t)]^{(2)} = \pi_\lambda [\hat{A}(\nabla^{TX} ch(\nabla^E))]^{(2)} + O(\sqrt{t})
\]

(3.9)

**Proposition 3.2.5.**

\[
[Tr_s \exp(-A^2_t)]^{(2)} = [Tr_s \exp(-\sqrt{t}D^X + \nabla^H)^2]^{(2)}
\]

(3.10)

**Proof.** Let $A^1_t = \sqrt{t}D^X + \nabla^H - \frac{c(T^H)}{4\sqrt{t}}$, then

\[
\frac{\partial}{\partial t}[Tr_s \exp(-A^2_t)]^{(2)} = -dTr_s \frac{\partial A^1_t}{\partial t} \exp(-A^2_t) = dTr_s \exp(-A^2_t) = 0
\]

Since $A^1_t = A_t$, $A^0_t = \sqrt{t}D^X + \nabla^H$, the proposition follows. \qed
Proposition 3.2.6.

\[ [Tr_s \exp(-\sqrt{t}D^X + \nabla H^2)]^{(2)} = [Tr_s \exp(-\sqrt{t}D^{\leq a} + \nabla H^{\leq a})]^2 + [Tr_s \exp(-\sqrt{t}D^{(a,\infty)} + \nabla H^{(a,\infty)})]^2 \]

(3.11)

Proof. We use the same transgression trick: Let \( M^a = \nabla H - \nabla H^{\text{split}} \), where \( \nabla H^{\text{split}} = \nabla H^{\leq a} \oplus \nabla H^{(a,\infty)} \), and \( \nabla H^t = \nabla H^{\text{split}} + IM^a \). Then

\[
\frac{\partial}{\partial t} [Tr_s \exp(-\sqrt{t}D^X + \nabla H^2)]^{(2)} = -dTr_s[M^a \exp(-\sqrt{t}D^X + \nabla H^2)]^{(1)}
\]

\[
= -dTr_s[M^a \exp(-tD^X,2)]
\]

Clearly \( M^a \) interchanges \( H^{\leq a} \) and \( H^{(a,\infty)} \), while \( \exp(-tD^X,2) \) preserves the splitting \( H_\pm = H^{\leq a} \oplus H^{(a,\infty)}_\pm \), so \( Tr_s[M^a \exp(-tD^X,2)] = 0 \). Since \( \nabla H^{\text{split}} \) verifies the proposition, so does \( \nabla H \). \( \square \)

Now

\[
\frac{\partial}{\partial t} [Tr_s \exp(-\sqrt{t}D^{(a,\infty)} + \nabla H^{(a,\infty)})^2]^{(2)} = -dTr_s[D^{(a,\infty)}]^{(2)} \exp(-\sqrt{t}D^{(a,\infty)} + \nabla H^{(a,\infty)})^2]^{(1)}
\]

\[
= \frac{1}{2} dTr_s[D^{(a,\infty)}]^{(2)} \exp(-tD^{(a,\infty),2})]
\]

\[
= \frac{1}{2} dTr_s[\exp(-tD^2)\nabla H \cdot D \cdot P^{(a,\infty)}]
\]

Since \( a > 0 \), as \( t \to +\infty \), \( [Tr_s \exp(-\sqrt{t}D^{(a,\infty)} + \nabla H^{(a,\infty)})^2]^{(2)} \) decays exponentially. So we have

\[
[Tr_s \exp(-\sqrt{t}D^{(a,\infty)} + \nabla H^{(a,\infty)})^2]^{(2)} = -\frac{1}{2} d \int_1^\infty Tr_s[\exp(-sD^2)\nabla H \cdot D \cdot P^{(a,\infty)}] ds
\]

\[
= -\frac{1}{2} d \delta_t^a
\]

(3.12)

By formula (3.9),(3.10),(3.11),(3.12), we have

\[
[Tr_s \exp(-A_1^2)]^{(2)} = [Tr_s \exp(-\sqrt{t}D^{\leq a} + \nabla H^{\leq a})^2]^{(2)} - \frac{1}{2} d \delta_t^a
\]

Let \( t \to 0 \), we get

\[
\pi_*[\hat{A}(\nabla^T \! X) ch(\nabla E)]^{(2)} = c_1(\nabla^{\leq a}) - \frac{1}{2 \pi i} \frac{1}{2} d \delta_0^a
\]

(3.13)

Now we have \( \nabla \nabla^{\leq a} = \nabla^{\leq a} + \frac{1}{2} (\gamma_0^a - \delta_0^a) + \frac{1}{2} \Gamma'(1) dA_0 \). Note that \( \gamma_0^a = -d[\frac{\partial c_1}{\partial s}(0)] - \Gamma'(1) dA_0 \) is exact, so

\[
c_1(\nabla^{\leq a}) = c_1(\nabla^{\leq a}) - \frac{1}{2 \pi i} \frac{1}{2} d \delta_0^a
\]

(3.14)

So \( \nabla^{\leq a} \) is the connection we want:

Theorem 3.2.2. Let \( \nabla^{\leq a} = \nabla^{\leq a} + \frac{1}{2} (\gamma_0^a - \delta_0^a) + \frac{1}{2} \Gamma'(1) dA_0 \), then identifying \( \lambda \) with \( \lambda_{\leq a} \) over \( U_a \), the connection \( \nabla^{\leq a} \) patch together into a connection \( \nabla^\lambda \) on \( \lambda \), which is unitary w.r.t. the metric \( \| \cdot \| \), and

\[
c_1(\nabla^\lambda) = \pi_*[\hat{A}(\nabla^T \! X) ch(\nabla E)]^{(2)}
\]

30