Homework 8: Part 2

1: Let $f(z)$ be a complex valued function, not necessarily analytic. Let $z = z(t)$ be a smooth arc. Denote $w(t) = f(z(t))$.

(a) Prove the following chain rule:

$$w'(t) = \frac{\partial f}{\partial z} z'(t) + \frac{\partial f}{\partial \bar{z}} \bar{z}'(t).$$  \hfill (1)

Note that in the above formula, we have used the following notation (compare Exercise 8 on Page 71)

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) f, \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f.$$

(b) If $f(z)$ is analytic, then by (Homework Exercise 8 on Page 71) we have $\frac{\partial f}{\partial \bar{z}} = 0$. Show that in this case, equation (1) is reduced to the following formula (compare Exercise 5 on Page 124):

$$w'(t) = \frac{\partial f}{\partial z} z'(t) = f'(z(t))z'(t).$$

2*: Show that $w = \sin(z)$ maps the vertical strip $\{z \in \mathbb{C}; -\frac{\pi}{2} < \text{Re}(z) < \frac{\pi}{2}\}$ to the region $\mathbb{C}\backslash((-\infty, -1] \cup [1, +\infty))$ by showing the following steps.

(a) Show that $\sin(z) = \sin x \cosh y + i \cos x \sinh y$.

(b) Use the identity $\cosh^2 y - \sinh^2 y = 1$ to show that $w = \sin(z)$ maps the vertical line $L_c = \{\text{Re}(z) = c\}$ to one branch of the following hyperbola:

$$\frac{x^2}{\sin^2(c)} - \frac{y^2}{\cos^2(c)} = 1.$$

If $c < \text{(resp. >)} 0$, then $L_c$ is mapped to the left (resp. right) branch. If $c = 0$, then $L_0$ is the $y$-axis, which is mapped to the $u$-axis in the $w$-plane ($w = u + iv$).
(c) As $c$ increases from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, visualize how the corresponding hyperbola moves continuously from the left to the right, and how the opening changes as follows:

ray $(-\infty, -1] \rightarrow$ small toward the left $\rightarrow$ large toward the left $\rightarrow$ vertical flat $\rightarrow$ large toward the right $\rightarrow$ small toward the right $\rightarrow$ ray $[1, +\infty)$. 