Newton’s law of cooling/heating

This law says that the rate of cooling/heating is proportional to the difference of the temperature of the object and cooling/heating source. Suppose the temperature of the cooling/heating source changes according to the function \( A(t) \). Let \( T = T(t) \) be the temperature of the object under consideration. Then we can write down the differential equation:

\[
\frac{dT}{dt} = -k(T - A(t)).
\]

\( k \) is called the cooling/heating constant. This is a 1st order linear differential equation:

\[
\frac{dT}{dt} + kT = kA(t).
\]

We can find the integrating factor:

\[
F(t) = e^{\int kdt} = e^{kt}.
\]

So it’s easy to find the solution:

\[
T(t) = e^{-kt} \int k e^{kt} A(t) dt.
\] (1)

**Example:** Let the cooling/heating constant be \( k = 0.3 \). Assume the initial temperature \( T(0) = -20^\circ C \). Assume the temperature of the source oscillates according to the function:

\[
A(t) = 10 \sin \left( \frac{\pi}{12} t \right).
\]

How does the temperature \( T(t) \) of the object change?

**Solution:** By the above discussion, we just need to calculate:

\[
T(t) = e^{-0.3t} \int 0.3 e^{0.3t} 10 \sin(\pi t/12) dt.
\] (2)

To simplify the calculation a little bit, we can use the substitution: \( u = \pi t/12 \), the right hand side becomes:

\[
T(u) = e^{-3.6u/\pi} \frac{36}{\pi} \int e^{3.6u/\pi} \sin(u) du
\] (3)

Now we can integrate by parts twice (let \( a = 3.6/\pi \))

\[
\int e^{au} \sin(u) du = \int e^{au}(-\cos u) = -e^{au} \cos u + a \int \cos(u)e^{au} du
\]

\[
= -e^{au} \cos u + a \int e^{au} \sin(u) = -e^{au} \cos u + ae^{au} \sin u - a^2 \int e^{au} \sin u
\]
So we can solve:

\[ \int e^{au} \sin(u) \, du = e^{au} \frac{a \sin u - \cos u}{1 + a^2} + C. \]

Now we can substitute in to (3) to get

\[ T(u) = \frac{36}{\pi} \left( \frac{a \sin u - \cos u}{1 + a^2} + Ce^{-au} \right) \]

When \( t = 0, u = 0, \) so we can use the initial condition \( T(0) = -20 \) to get

\[-20 = T(0) = \frac{36}{\pi} \left( -\frac{1}{1 + a^2} + C \right) \implies C = \frac{1}{1 + a^2} - \frac{5\pi}{9}. \]

Substitute \( u = \pi t/12 \) and \( a = 3.6/\pi \) \((au = 0.3t)\), then finally we get the particular solution:

\[ T(t) = \frac{36}{\pi} \left( \frac{3.6 \sin \frac{\pi t}{12} - \cos \frac{\pi t}{12}}{1 + (3.6/\pi)^2} + \left( \frac{1}{1 + (3.6/\pi)^2} - \frac{5\pi}{9} \right) e^{-0.3t} \right) \]

\[ \approx 5.68 \sin(0.26t) - 4.95 \cos(0.26t) - 15.05 e^{-0.3t}. \]

The term \(-15.05 e^{-0.3t}\) in the solution is called the “damped” part. The following is the plot of graphs using Mathematica. The blue curve is the temperature of the source, and the red curve is the temperature of the object. Also the first command \((T1[t_] = \ldots)\) defines the function \(T1(t)\) to be plotted.

From the graphs, we see that the temperature of the object will also oscillate in the long term. But the amplitude of oscillation (red curve) is smaller than the amplitude of the source (blue curve). The oscillation of the temperature of the object lags behind the oscillation of the source.
We can actually calculate by using Mathematica. For this rewrite equality (2) such that it satisfies the initial condition:

\[ T(t) = e^{-0.3t} \left( \int_0^t 0.3e^{0.3\tau} 10\sin(\pi\tau/12)d\tau - 20 \right). \]

Then we can use Mathematica to integrate and plot the graphs: