Population models

Notation:
- \( P = P(t) \) population at time \( t \);
- \( \beta = \beta(P, t) \) birth rate;
- \( \delta = \delta(P, t) \) death rate. The most general form of differential equation modeling the population is:

\[
\frac{dP}{dt} = (\beta(P, t) - \delta(P, t))P(t), \quad P(0) = P_0.
\] (1)

Model 1: Logistic model
In this model, \( \beta = \beta_0 - kP \), \( \delta = \delta_0 \). \( \beta_0, k, \delta_0 \) are constants. So the equation (1) becomes:

\[
\frac{dP}{dt} = (\beta_0 - kP - \delta_0)P = kP(M - P), \quad P(0) = P_0.
\] (2)

Here \( M = (\beta_0 - \delta_0)/k \). Equation (2) is called a logistic equation. It’s a separable equation:

\[
\frac{1}{M} \left( \frac{1}{P} + \frac{1}{M - P} \right) dP = \frac{dP}{P(M - P)} = kdt
\]

So if we integrate on both sides, we get:

\[
\frac{1}{M} \ln \left( \frac{P}{M - P} \right) = kt + C_1 \implies \frac{P}{M - P} = e^{MC_1} e^{kMt} = Ce^{kMt}.
\]

Here \( C = e^{MC_1} \) is a positive constant. We can determine it using the initial condition \( P(0) = P_0 \):

\[
C = \frac{P_0}{M - P_0}.
\]

Now we can solve \( P = P(t) \) to get a general solution:

\[
P(t) = \frac{MCe^{kMt}}{1 + Ce^{kMt}} = \frac{M}{1 + C^{-1}e^{-kMt}} = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}.
\] (3)

From the solution (3), we see that
- For any initial population \( P_0 > 0 \), we always have
  \[
  \lim_{t \to +\infty} P(t) = M.
  \]
  \( M \) is called the carrying capacity in this logistic model.
- \( P(t) \equiv M \) is a solution. This solution is called an equilibrium solution. It is a stable equilibrium.
We can use Mathematica to draw solution curves. For simplicity, assume $k = 1, M = 2$.

**Model 2: Doomsday-Extinction model** In this model, $\beta = kP, \delta = \delta_0$ with $k, \delta_0$ constants. So the equation (1) becomes

$$\frac{dP}{dt} = (kP - \delta)P = k(P - M)P, \quad P(0) = P_0. \tag{4}$$

Here $M = \delta/k$. Again this is a separable equation:

$$\frac{1}{M} \left( \frac{1}{P - M} - \frac{1}{P} \right) = \frac{dP}{(P - M)P} = kdt.$$

So we integrate both sides to get:

$$\frac{1}{M} \log \frac{P - M}{P} = kt + C_1 \implies \frac{P - M}{P} = Ce^{Mkt}.$$  

Here $C = e^{MC_1}$ is a positive constant. Substituting $P(0) = P_0$ we can determine $C$:

$$C = \frac{P_0 - M}{P_0}.$$  

So we can solve $P = P(t)$ to get:

$$P(t) = \frac{M}{1 - Ce^{Mkt}} = \frac{MP_0}{P_0 - (P_0 - M)e^{Mkt}}. \tag{5}$$

From the solution (5) we see that:
• If $P_0 > M$, then the population will explode to infinity at time when the denominator becomes 0:

$$t_{\text{doom}} = \frac{1}{Mk} \log \frac{P_0}{P_0 - M}.$$

• If $P_0 < M$, then the population will decay exponentially to 0.

$$\lim_{t \to +\infty} P(t) = 0.$$

• $P(t) \equiv M$ is an (unstable) equilibrium solution.

Again we can use Mathematica to draw some streamlines to visualize the situation ($k = 1, M = 2$).