Final Review Sheets

1 Basic concepts

1. n-th order DE, existence and uniqueness, direction fields, general solutions, particular solutions, initial value problem, autonomous equation, equilibrium solution, stability of equilibrium solution

2. linear DE, homogeneous linear DE, non-homogeneous linear DE, basic solutions, complementary solutions, structure of general solution.

3. transformation to 1st order system, linear system, eigenvalues/eigenvectors, generalized eigenvectors, fundamental solution matrix, exponential of matrices

4. series solutions, radius of convergence, recurrence relations

2 1st order equation

Method: (a) Separable equations

\[ \frac{dy}{dx} = \frac{1 + \sqrt{y}}{1 + \sqrt{x}}. \]

(b) Exact equation

\[ (x^2 + \ln y)dx + \left(y^3 + \frac{x}{y}\right) dy = 0. \]

(c) Linear 1st order equation (integrating factor)

\[ (1 - x^2)y' + xy = 1. \]

(d) Substitution

• (Homogeneous)

\[ y \frac{dy}{dx} - y = \sqrt{x^2 + y^2}. \]

• (Bernoulli type)

\[ (1 + x) \frac{dy}{dx} + y = y^3. \]

Models: (a) Newton’s Law of cooling: Notes, HW2 (6)

(b) Population model: Notes, HW4 (2,3), Practice1 (4), Mid1 (4).

(c) Acceleration-velocity model: HW5 (1), Practice1 (5)
3 Linear DE with constant coefficients

Method: (a) Homogeneous linear equations

\[ y''' - 2y'' + y = 0. \]

(b) Non-homogeneous equations, particular solutions
   i. Undetermined coefficients
   \[ y'' + 4y' + 4y = e^{-2x}. \]
   ii. Variation of parameters
   \[ y'' + y = \frac{1}{\sin^2 x}. \]

Model: Mechanical vibration HW7 (9-11), HW9 (1-2), Practice2 (3), Midterm 2 (3)-(4).

(a) Free undamped/damped oscillation
(b) Forced undamped/damped oscillation, Resonance

4 Linear System

Method: (a) Elimination method HW10 (1)-(2), Quiz3 (1), HW11 (3a) (also the following three systems).
(b) Eigenvalue method
\[
\begin{align*}
    x_1' &= 4x_1 + 2x_2 \\
    x_2' &= -3x_1 - x_2 \\
    x_3' &= x_1 + x_2 + 2x_3
\end{align*}
\]
(c) Complex eigenvalue
\[
\begin{align*}
    x_1' &= 7x_1 + x_2 \\
    x_2' &= -4x_1 + 3x_2
\end{align*}
\]
(d) Multiple eigenvalues
\[
\begin{align*}
    x_1' &= x_1 - 4x_2 \\
    x_2' &= 4x_1 + 9x_2
\end{align*}
\]
(e) Chain of generalized eigenvectors: Notes, HW11 (3b), HW12 (1).
(f) Exponential of matrices and initial value problems: HW12 (2), (3).

(g) Nonhomogeneous linear system

\[
\begin{align*}
    x'_1 &= 2x_1 + 3x_2 + e^t \\
    x'_2 &= 2x_1 + x_2 + e^{2t}
\end{align*}
\]

Model: (2-mass, 3-spring) system: HW9 (4), HW10 (2).

5 Power Series Solutions

Standard Maclaurin series, Radius of convergence, series solutions, recurrence relations

1st order: \((2x + 1)y' = y\).

2nd order: \(y'' - 2xy' + 6y = 0\).