Homework 4

1. First solve the equation \( f(x) = 0 \) to find the critical points of the given autonomous differential equation. Then determine whether each critical point is stable or unstable, and construct the corresponding phase diagram for the differential equation. Next, solve the differential equation explicitly for \( x(t) \). Finally, use either the exact solution or computer generated solution curves to verify the stability.

(a) \( x' = x^2 - 4x + 3 \).
(b) \( x' = x^2 - 4x + 5 \).
(c) \( x' = -x^2 + 2x - 1 \).
(d) \( x' = -x^2 + 2x + 3 \).

2. The differential equation \( \frac{dx}{dt} = \frac{1}{8}x(8 - x) - h \) models a logistic population with harvesting at rate \( h \). Determine the dependence of the number of critical points on the parameter \( h \), and then construct a bifurcation diagram in the \( hc \)-plane. Use \texttt{Manipulate} in \texttt{Mathematica} to visualize the bifurcation process.

3. The differential equation \( \frac{dx}{dt} = \frac{1}{8}x(x - 8) + s \) models a explosion/extinction population with stocking at rate \( s \). Determine the dependence of the number of critical points \( c \) on the parameter \( s \) and then construct a bifurcation diagram in the \( sc \)-plane. Use \texttt{Manipulate} in \texttt{Mathematica} to visualize the bifurcation process.

4. A woman bails out of an airplane, falls freely for 20 s, then opens her parachute. Assume the drag coefficient \( \rho = 0.15 \) without parachute and \( \rho = 1.5 \) with the parachute. Find the velocity \( v(t) \) as a function of \( t \) and terminal velocity in the following two situations:

(a) Assume the air resistance is \( \rho v \) ft/s/2.
(b) Assume air resistance is \( \rho v^2 \) ft/s/2.

Compare the two velocity functions by plotting their graphs (using \texttt{Mathematica}).

5. (*) (Taking account of the moon’s gravitational field, re-solve the problem 1.2.42 in homework 1) A spacecraft is in free fall toward the surface of the moon at a speed of 1000 \( mi/h \). Its retrorocket, when fired, provide a constant deceleration of 20,000 \( mi/h^2 \). At what height above the lunar surface should the astronauts fire the retrorockets to insure a soft touchdown? Note that you need to change the units by using the following data.
\[ G \approx 6.6726 \times 10^{-11} \text{N} \cdot (m/\text{kg})^2, \quad M_{\text{moon}} = 7.35 \times 10^{22} \text{(kg)}, \quad R_{\text{moon}} = 1740 \text{km}. \]

6. (*) To what radius would the moon have to be compressed in order for it to become a black hole - the escape velocity from its surface equal to the velocity \( c = 3 \times 10^8 \text{m/s} \) of light?