THERE ARE EIGHT (8) PROBLEMS. THEY HAVE THE INDICATED VALUE.

SHOW YOUR WORK

DO NOT TEAR-OFF ANY PAGE

NO CALCULATORS NO CELLS ETC.

ON YOUR DESK: ONLY test, pen, pencil, eraser.

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1. (50pts) Linear system/Gauss-Jordan

Solve the linear system:

\[
\begin{align*}
    x_1 + x_2 + x_3 + x_4 &= 1 \\
    x_1 + x_2 + 2x_3 + 4x_4 &= 4 \\
    -2x_1 - 2x_2 - x_3 + x_4 &= 1
\end{align*}
\]
2. (50pts) Matrix operation: transpose/product/inverse

\[ A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 2 & 1 & 2 \end{pmatrix}. \]

(a.) Calculate \( A^T A \).

(b.) Find the \( A^{-1} \). Express the inverse of matrices in part (a) using \( A^{-1} \). You don’t need to calculate.
3. (50pts) Basis/orthonormal basis/orthogonal complement
Consider the plane in \( \mathbb{R}^3 \) defined by:
\[
V : x - 2y + 2z = 0.
\]
(a.) Find a basis \( B_1 \) of \( V \).
(b.) Use Gram-Schmidt process to get an orthonormal basis \( B_2 \) of \( V \).
(c.) Find another vector \( \vec{v}_3 \in \mathbb{R}^3 \) such that \( B_2 \cup \{ \vec{v}_3 \} \) is an orthonormal basis of \( \mathbb{R}^3 \).
4. (50pts) Linear transformation/Linear isomorphism
Consider the transformation $T : P_2 \rightarrow P_2$. $T(f) = x^2 f'' - xf' + f$.

(a.) Is $T$ linear? Why?
(b.) Is $T$ an isomorphism?
5: Orthogonal matrices/Properties of orthogonal matrices

(a.) Find the matrix $A$ representing the rotation around $\vec{e}_3 -$axis by $30^{\circ}$, counterclockwise as viewed from positive $\vec{e}_3$-axis.

(b.) Find the matrix $B$ representing the rotation around $\vec{e}_1 -$axis by $90^{\circ}$, counterclockwise as viewed from positive $\vec{e}_1$-axis.

(c.) Calculate $AB$. Is it an orthogonal matrix? Why?
6. (50pts) Determinant/Adjoint matrices and its relation with inverses

Let

\[ A = \begin{pmatrix}
1 & 2 & 3 & 4 \\
1 & 0 & 2 & 4 \\
3 & 0 & 2 & 0 \\
2 & 0 & 0 & 0
\end{pmatrix}. \]

(a.) Calculate the determinant of \( A \)

(b.) Calculate the (23)-entry of \( \text{adj}(A) \).
7. (50pts) Matrices of orthogonal projections/Diagonalizability

Consider the line $L = \text{Span}\{(2, 4, 5)\}$ in $\mathbb{R}^3$ passing through the origin.

(a.) Find the matrix representing the projection to $L$ in $\mathbb{R}^3$.

(b.) Is the above obtained matrix diagonalizable? If it is, what is the corresponding diagonal matrix?
8. (50pts) Eigenvalue/Eigenvector/Eigenspaces

Let

\[ A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}. \]

(a.) Find eigenvalues of \( A \).
(b.) Determine eigenspaces of \( A \).
(c.) Is \( A \) diagonalizable? If it is, then what’s the corresponding diagonal matrix?
Scratch paper