(6\textsuperscript{10} means problem 6 in 10-th edition textbook, 6\textsuperscript{9} means problem 6 in 9-th edition textbook)

13.8

\(6\textsuperscript{10} = 6\textsuperscript{9}:\) \(f(x, y) = -x^2 - y^2 + 10x + 12y - 64 = -(x - 5)^2 - (y - 6)^2 - 3.\) So \(f(5, 6) = -3\) is a absolute maximum. Use partial derivative to verify:

\[f_x = -2x + 10 = 0, f_y = -2y + 12 = 0 \implies x = 5, y = 6.\]

By 2nd partial test the critical point \((5, 6)\) is a relative maximum.

\(12\textsuperscript{10} = 10\textsuperscript{9}:\) \(f(x, y) = 2x^2 + 2xy + y^2 + 2x - 3.\)

\[f_x = 4x + 2y + 2 = 0, f_y = 2x + 2y = 0 \implies (x, y) = (-1, 1).\]

So the critical point \((-1, 1)\) is a relative minimum.

13.9

\(6\textsuperscript{10} = 6\textsuperscript{9}:\) \(x + y + z = 32.\) Maximize \(P = xy^2z.\) We can solve \(z = 32 - x - y.\)

So \(P(x, y) = xy^2(32 - x - y).\) We find critical point by setting:

\[P_x = 32y^2 - 2xy^2 - y^3 = y^2(32 - 2x - y) = 0, P_y = 64xy - 2x^2y - 3xy^2 = xy(64 - 2x - 3y) = 0.\]

We get the following critical points:

\((x, 0)\) for any \(x;\) \((0, 32);\) \((8, 16).\)

The maximum is obtained when \((x, y) = (8, 16)\) for which \(z = 32 - x - y = 8.\)

\(P_{max} = 8 \times 16^2 \times 8 = 16384.\)

\(10\textsuperscript{10} = 10\textsuperscript{9}:\) If the length, width and height of the box are denoted by \(x, y, z.\) Then the cost for constructing the box is: \(1.5xy + 2(xz + yz).\) To find the maximum volume we can assume \(1.5xy + 2(xz + yz) = C.\) Under this constraint, we want to maximize \(V = xyz.\) We can solve \(z = (C - 1.5xy)/(2(x + y)).\) Then we get

\[V(x, y) = xy \cdot \frac{C - 1.5xy}{2(x + y)}.\]

We find critical point (maximum) by solving:

\[V_x = \frac{(Cy - 3xy^2)(x + y) - (Cxy - 1.5x^2y^2)}{2(x + y)^2} = 0,\]
and
\[ V_y = \frac{(C + 3x^2)(x + y) - (Cxy - 1.5x^2y^2)}{2(x + y)^2} = 0. \]

We can assume \( x > 0 \) and \( y > 0 \), then, after some calculations, these equations simply to
\[ 1.5x^2 + 3xy = C, \quad 1.5y^2 + 3xy = C. \]

We can solve these two equations to get \( x = y = \sqrt{C/4.5} = \sqrt{2C/3} \). The height is
\[ z = \frac{C - 1.5xy}{x + y} = \frac{\sqrt{C}}{2\sqrt{2}}. \]

14\(^{10} = 14^9\): The profit
\[ P(x_1, x_2) = 15(x_1 + x_2) - (0.02x_1^2 + 4x_1 + 500) - (0.05x_2^2 + 4x_2 + 275). \]

We find maximum using vanishing of partial derivatives:
\[ P_{x_1} = 15 - 0.04x_1 - 4 = 0, \quad P_{x_2} = 15 - 0.1x_2 - 4 = 0 \Rightarrow (x_1, x_2) = (275, 110). \]

18\(^{10} = 16^9\): The area equals
\[ A(x, \theta) = (30 - 2x + x \cos \theta) \cdot x \sin \theta = 30x \sin \theta - 2x^2 \sin \theta + x^2 \sin \theta \cos \theta. \]

We find the maximum by setting:
\[ A_x = (30 - 4x + 2x \cos \theta) \sin \theta = 0, \quad A_\theta = x(30 \cos \theta - 2x \cos \theta + x(\cos^2 \theta - \sin^2 \theta)) = 0. \]

From the first equation, we get \( \cos \theta = (2x - 15)/x \) since we can assume \( \sin \theta \neq 0 \). Substituting \( \cos \theta \) into the 2nd equation and using the relation \( \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 \), we get the equation for \( x \):
\[ 30 \frac{2x - 15}{x} - 2x \frac{2x - 15}{x} + x \left( 2 \left( \frac{2x - 15}{x} \right)^2 - 1 \right) = 0. \]

Simplifying this equation, we get: \( 3x(x - 10) = 0 \). So \( x = 10 \) and \( \cos \theta = 1/2 \). So \( \theta = \pi/3 = 60^\circ \).

3 13.10

6\(^{10} = 8^9\): \( f(x, y) = 3x + y + 10. \) Constraint \( g(x, y) = x^2y - 6 = 0 \). By Lagrange Multiplier method, we get three equations:
\[ \begin{cases} \nabla f = \lambda \nabla g \\ x^2y - 6 = 0. \end{cases} \]

\[ \begin{cases} 3 = \lambda 2xy \\ 1 = \lambda x^2 \\ x^2y = 6. \end{cases} \]

Canceling \( \lambda \) from the first 2 equations, we get \( y = 3x/2 \). Substitute this into the last equation we get \( 3x^3/2 = 6 \). So we get \( x = 4^{1/3} \). From this we get \( y = 1.5 \times 4^{1/3} \) and \( \lambda = 4^{-2/3} \). So the maximum of \( f(x, y) \) under the constraint is equal to
\[ f(4^{1/3}, 1.5 \times 4^{1/3}) = 4.5 \times 4^{1/3} + 10. \]
2410 = 269: Minimize the squared-distance function \( f(x, y) = x^2 + (y - 10)^2 \) under the constraint \( g(x, y) = (x - 4)^2 + y^2 - 4 = 0 \). Using Lagrange Multiplier method, we get three equations:

\[
\begin{align*}
2x &= \lambda (x - 4) \\
2(y - 10) &= \lambda 2y \\
(x - 4)^2 + y^2 &= 4.
\end{align*}
\]

From the 1st equation, we get \( x = \frac{4\lambda}{\lambda - 1} \). From then 2nd equation, we get \( y = -\frac{10}{\lambda - 1} \). Substitute these into the last equation, we get an equation for \( \lambda \):

\[
\left( \frac{4\lambda}{\lambda - 1} - 4 \right)^2 + \frac{100}{(\lambda - 1)^2} = 4 \iff (\lambda - 1)^2 = 29.
\]

So we get two solutions: \( \lambda = 1 \pm \sqrt{29} \).

- If \( \lambda = 1 + \sqrt{29} \), \( x = 4(1 + \sqrt{29})/\sqrt{29} \) and \( y = -10/\sqrt{29} \). The squared-distance is

\[
f \left( \frac{4(\sqrt{29} + 1)}{\sqrt{29}}, \frac{-10}{\sqrt{29}} \right) = \frac{16(\sqrt{29} + 1)^2 + 100(\sqrt{29} + 1)^2}{29} = 4(\sqrt{29} + 1)^2.
\]

- If \( \lambda = 1 - \sqrt{29} \), \( x = 4(\sqrt{29} - 1)/\sqrt{29} \) and \( y = 10/\sqrt{29} \). The squared-distance is:

\[
f \left( \frac{4(\sqrt{29} - 1)}{\sqrt{29}}, \frac{10}{\sqrt{29}} \right) = \frac{16(\sqrt{29} - 1)^2 + 100(\sqrt{29} - 1)^2}{29} = 4(\sqrt{29} - 1)^2.
\]

The 2nd value is smaller, so the minimal distance is equal to \( 2(\sqrt{29} - 1) \).

**Remark 3.1** Note that one can solve the problem using geometry. The closest point on the circle to a point (outside of the circle) is the intersection point of the circle and line segment connecting the point and the center.

3410 = 369: We want to maximize \( P(x, y, z) = xyz^2 \) under the constraint \( g(x, y, z) = x + y + z - 32 = 0 \). Using Lagrange Multiplier method, we get equations:

\[
\begin{align*}
\nabla P &= \lambda \nabla g \\
g(x, y, z) &= 0.
\end{align*}
\]

To find the maximum value of \( P \), we can assume \( x, y, z \) are nonzero. So \( \lambda \neq 0 \) for the solution. Divide the first 2 equations, we get \( y = 2x \). Divide the 2nd and 3rd equation, we get \( y = 2z \). So \( x = z = y/2 \). Substitute this into the last equation, we get \( 2y = 32 \). So \( y = 16 \) and \( x = z = 8 \). The maximum value of \( P \) is \( P(8, 16, 8) = 16384 \).

4610 = 509: We assume the area is fixed, so \( lh + \pi(l/2)^2/2 = A \) is fixed. In other words, we have the constraint \( g(l, h) = l \cdot h + \pi l^2/8 - A = 0 \). We want to
minimize the diameter \( f(l, h) = \pi \cdot l/2 + l + 2h \). Using the Lagrange Multiplier, we get the equation:

\[
\begin{align*}
\pi/2 + 1 &= \lambda(h + \pi l/4) \\
2 &= \lambda l \\
 lh + \pi l^2/8 &= A.
\end{align*}
\]

From the 2nd equation, we get \( l = 2/\lambda \). Substitute this into first equation, we solve \( h = 1/\lambda \). So \( l = 2h \), i.e. the length of the rectangle is twice its height.