

(2^{10} \text{ means problem 24 in 10-th edition textbook, } 14^{9} \text{ means problem 24 in 9-th edition textbook })

1 13.6

2^{10} = 14^{9}: f(x, y) = y/(x + y), P(3, 0), \theta = -\pi/6.

\[ f_x(3, 0) = \left. \frac{y}{(x + y)^2} \right|_{(3,0)} = 0, \quad f_y = \left. \frac{x}{(x + y)^2} \right|_{(3,0)} = \frac{1}{3}. \]

So

\[ D_uf(3, 0) = f_x \cos \left( -\frac{\pi}{6} \right) + f_y \sin \left( -\frac{\pi}{6} \right) = \frac{1}{3} \cdot \left( -\frac{1}{2} \right) = -\frac{1}{6}. \]

2^{14} = 28^{9}: f(x, y) = 3x^2 - y^2 + 4. f_x = 6x, f_y = -2y. \n\n\n\[ \nabla f(-1, 4) = (-6, -8). \]

\[ \frac{\overrightarrow{PQ}}{||\overrightarrow{PQ}||} = \frac{(4, 2)}{\sqrt{20}} = \frac{(2, 1)}{\sqrt{5}}. \]

The directional derivative is:

\[ (D_u f)(-1, 4) = \nabla f(-1, 4) \cdot \vec{u} = -\frac{20}{\sqrt{5}} = -4\sqrt{5}. \]

36^{10} = 40^{9}: f(x, y, z) = xe^{yz}. P(2, 0, -4).

\[ (\nabla f)(2, 0, -4) = (e^{yz}, xe^{yz}, xye^{yz})(2, 0, -4) = (1, -8, 0). \]

The maximum value of the directional derivative is:

\[ (D_{\nabla f} f)(2, 0, 4) = ||\nabla f|| (2, 0, -4) = ||(1, -8, 0)|| = \sqrt{65}. \]

52^{10} = 56^{9}: f(x, y) = x - y^2. c = 4. P(4, -1). (a): \n\n\[ (\nabla f)(4, -1) = \langle 1, -2y \rangle(4, -1) = \langle 1, 2 \rangle. \]

(b): Unit normal vector \( \vec{n} = (\nabla f)/||\nabla f|| = (1, 2)/\sqrt{5}. \)

(c): Tangent line equation:

\[ \langle 1, 2 \rangle \cdot \langle x - 4, y + 1 \rangle = 0 \iff x + 2y = 2. \]

(d): See Figure 1.

61^{10} = 66^{9}: h(x, y) = 5000 - 0.001x^2 - 0.004y^2. \n\n\n\[ \nabla h = \langle -0.002x, -0.008y \rangle. \]

At point (500, 300, 4390) the gradient is

\[ (\nabla h)(500, 300) = \langle -1, -2.4 \rangle. \]

The direction to get greatest rate of ascension is the direction of gradient vector:

\[ \vec{u} = \frac{\nabla h}{||\nabla h||} = \frac{\langle -1, -2.4 \rangle}{\sqrt{6.76}} \approx \langle -0.38, -0.92 \rangle. \]

1
2 13.7

$10^{10} = 18^9$: $f(x,y) = \frac{y}{x}$. (1, 2, 2) The graph is also a level surface $g(x, y, z) = \frac{y}{x} - z = 0$. So the normal vector is given by the gradient of function $$(\nabla g)(1, 2, 2) = \left(-\frac{y}{x^2}, \frac{1}{x}, -1\right)(1, 2, 2) = (-2, 1, -1).$$

So the tangent plane at (1, 2, 2) is:

$$(-2, 1, -1) \cdot (x-1, y-2, z-2) = 0 \iff 2x - y + z = 2.$$ 

$18^{10} = 28^9$: $x^2 + 2z^2 - y^2 = 0$. The gradient gives the normal vector $\langle 2x, -2y, 4z \rangle(1, 3, -2) = \langle 2, -6, -8 \rangle$. So the tangent plane is:

$$2(x-1) - 6(y-3) - 8(z+2) = 0 \iff x - 3y - 4z = 0.$$ 

$24^{10} = 34^9$: $z = 16 - x^2 - y^2$. (2, 2, 8). Write this as the level set: $x^2 + y^2 + z - 16 = 0$. The normal vector is given by the gradient:

$$\langle 2x, 2y, 1 \rangle(2, 2, 8) = \langle 4, 4, 1 \rangle.$$ 

The tangent plane is:

$$4(x-2) + 4(y-2) + (z-8) = 0 \iff 4x + 4y + z = 24.$$ 

The set of symmetric equations for the normal lines at (2, 2, 8) is:

$$\frac{x-2}{4} = \frac{y-2}{4} = \frac{z-8}{1}.$$ 

$42^{10} = 52^9$: $z = 3x^2 + 2y^2 - 3x + 4y - 5$. The tangent plane is horizontal if and only if the partial derivatives both vanish.

$$\frac{\partial z}{\partial x} = 6x - 3 = 0, \quad \frac{\partial z}{\partial y} = 4y + 4 = 0 \implies (x, y) = \left(\frac{1}{2}, -1\right).$$
18^10 = 26^9: \( f(x, y) = 2xy - \frac{1}{2}(x^4 + y^4) + 1 \). We set the partial derivatives to be 0. \( f_x = 2y - 2x^3 = 0, \ f_y = 2x - 2y^3 = 0 \). From the 1st equation we get \( y = x^3 \). Substitute this into the 2nd equation, we get

\[
0 = 2x - 2x^9 = 2x(1 - x^8) = 2x(1-x^4)(1+x^2)(1+x^4).
\]

So \( x = 0, 1, -1 \) and we get 3 critical points: \( P_1(0,0), P_2(1,1), P_3(-1,-1) \).

To classify the critical points, we calculate 2nd partial derivatives:

\[
f_{xx} = -6x^2, \ f_{xy} = 2, \ f_{yy} = -6y^2.
\]

\[
d = f_{xx}f_{yy} - f_{xy}^2 = 36x^2y^2 - 4.
\]

1. \( P_1(0,0) \): \( d = -4 < 0 \). This is a saddle point.
2. \( P_2(1,1) \): \( d = 32 > 0, \ f_{xx} = -6 < 0 \). This is a relative maximum.
3. \( P_3(-1,-1) \): \( d = 32 > 0, \ f_{xx} = -6 < 0 \). This is a relative maximum.

27 - 30^10 = 31 - 34^9:

27: \( d = f_{xx}f_{yy} - f_{xy}^2 = 9 \times 4 - 6^2 = 0 \). Inconclusive.

28: \( d = 20 > 0, \ f_{xx} = -3 < 0 \). This is a relative maximum.

29: \( d = -154 < 0 \). Saddle point.

30: \( d = 100 > 0, \ f_{xx} = 25 > 0 \). Relative minimum.

42^10 = 46^9: \( f(x, y) = x^2 + xy \). \( R = \{(x, y) : |x| \leq 2, |y| \leq 1\} \).

(a) Regions and candidate points

(b) Graph of the function

1. Find critical point in the interior:

\[
f_x = 2x + y = 0, \quad f_y = x = 0 \implies (x, y) = (0, 0).
\]

The 2nd partial test: \( f_{xx}f_{yy} - f_{xy}^2 = 2 \cdot 0 - 1^2 = -1 < 0 \). (0,0) is a saddle point. So it does not obtain either maximum or minimum.

2. Restrict \( f(x, y) \) to the boundary. There are four parts of the boundary:
• \( C_1 = \{ x = 2, |y| \leq 1 \} \). \( g_1(y) = f(2, y) = 4 + 2y \) on the interval \([-1, 1]\). \( \partial g_1 / \partial y = 2 \neq 0 \). There is no critical point in the interior of interval. We need to consider the end points: \((2, -1), (2, 1)\). The values at these points are:

\[
f(2, -1) = g_1(-1) = 2, \quad f(2, 1) = g_1(1) = 6.
\]

• \( C_2 = \{ y = 1, |x| \leq 2 \} \). \( g_2(x) = f(x, 1) = x^2 + x \) on the interval \([-2, 2]\). \( \partial g_2 / \partial x = 2x + 1 \). So there is a critical point for \( g_2 \) at \( x = -1/2 \). So we get two more candidate points: \((-1/2, 1)\) and \((-2, 1)\)(end point). The values at these points are:

\[
f(-1/2, 1) = g_2(-1/2) = -\frac{1}{4}, \quad f(-2, 1) = g_2(-2) = 2.
\]

• \( C_3 = \{ x = -2, |y| \leq 1 \} \). \( g_3(y) = f(-2, y) = 4 - 2y \) on the interval \([-1, 1]\). \( \partial g_3 / \partial y = -2 \neq 0 \). There is no critical point in the interior of interval. We need to consider the new end point: \((-2, -1)\). The value at this point is:

\[
f(-2, -1) = g_3(-1) = 6.
\]

• \( C_4 = \{ y = -1, |x| \leq 2 \} \). \( g_4(x) = f(x, -1) = x^2 - x \) on the interval \([-2, 2]\). \( \partial g_4 / \partial x = 2x - 1 \). So there is a critical point for \( g_4 \) at \( x = 1/2 \). So we get the last candidate point: \((1/2, -1)\). (The end points are already counted in) The value at this point is:

\[
f(1/2, -1) = g_4(1/2) = -\frac{1}{4}.
\]

3. By comparison, we get the 2 absolute maximum points: \( f(2, 1) = f(-2, -1) = 6 \), and 2 absolute minimum points: \( f(-1/2, 1) = f(1/2, -1) = -1/4 \).

4610 = 509: \( f(x, y) = 2x - 2xy + y^2 \). \( R = \{(x, y) : x^2 \leq y \leq 1\} \).

![Regions and candidate points](c) ![Graph of the function](d)

1. Find critical point in the interior:

\[
f_x = 2 - 2y = 0, \quad f_y = -2x + 2y = 0 \implies (x, y) = (1, 1).
\]

This point is on the boundary.
2. Restrict $f(x,y)$ to the boundary. There are 2 parts of the boundary:

- $C_1 = \{ y = x^2, |x| \leq 1 \}$. $g_1(x) = f(x,x^2) = 2x - 2x^3 + x^4$ on the interval $[-1,1]$. $\partial g_1/\partial x = 2 - 6x^2 + 4x^3 = 2(1 - 3x^2 + 2x^3)$. We factorize to find the critical point:

$$1 - 3x^2 + 2x^3 = 1 - x^2 - 2x^2 + 2x^3 = (1 - x)(1 + x - 2x^2(1 - x)$$
$$=(x - 1)(2x^2 - x - 1) = (x - 1)(x - 1)(2x + 1).$$

So we get 2 critical potions for $g_1(x)$: $x = 1, -1/2$. We get 3 candidate points: $(-1/2,1/4)$, $(1,1)$, $(-1,1)$. The values at these points are:

$$f(-1/2,1/4) = g_1(-1/2) = -\frac{11}{16}, f(1,1) = g_1(1) = 1, f(-1,1) = 1.$$

- $C_2 = \{ y = 1, |x| \leq 1 \}$. $g_2(x) = f(x,1) = 1$ on the interval $[-1,1]$. So $f$ is a constant along $C_2$. All the points on this line segment are candidate points.

3. By comparison, we get the absolute minimum: $f(-1/2,1/4) = -\frac{11}{16}$, and $f$ obtains absolute maximum value 1 at every point on the line segment $C_2 = \{ y = 1, |x| \leq 1 \}$. 
