Geodesic Zippers

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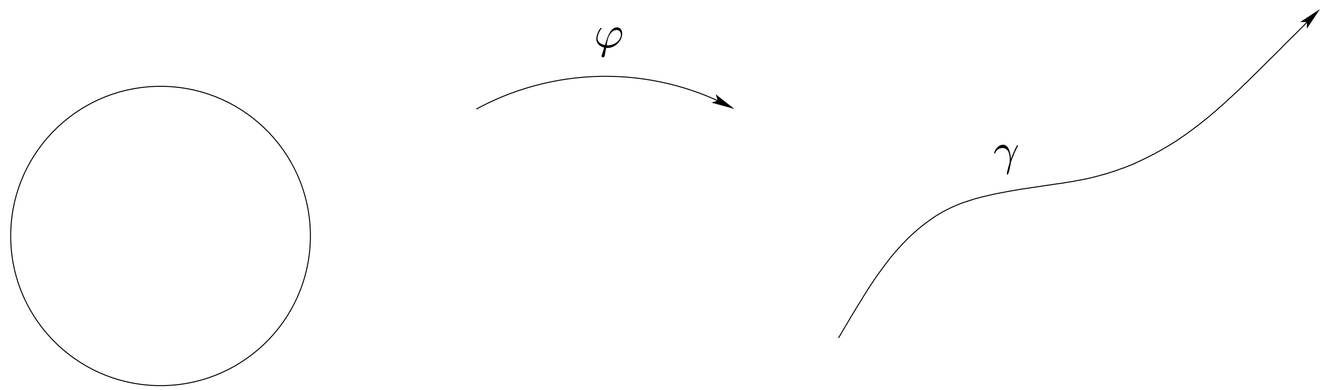
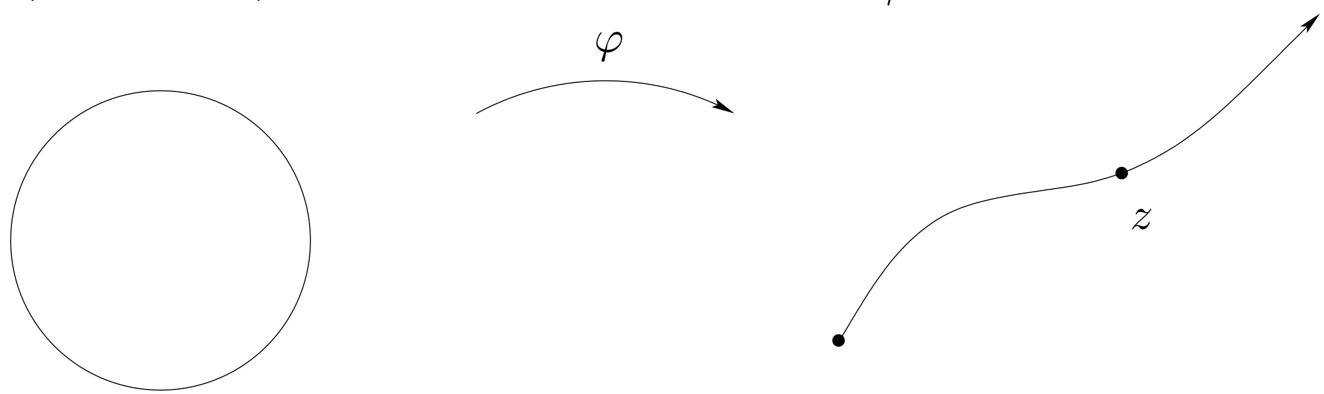
$$\sup_{\varphi}\left|\int \varphi K \, d\theta\right|$$



$$\sup_{\varphi}\left|\int \varphi K\, d\theta\right| \qquad \int_{\gamma} z\widetilde{K}(\omega)\, d\omega$$



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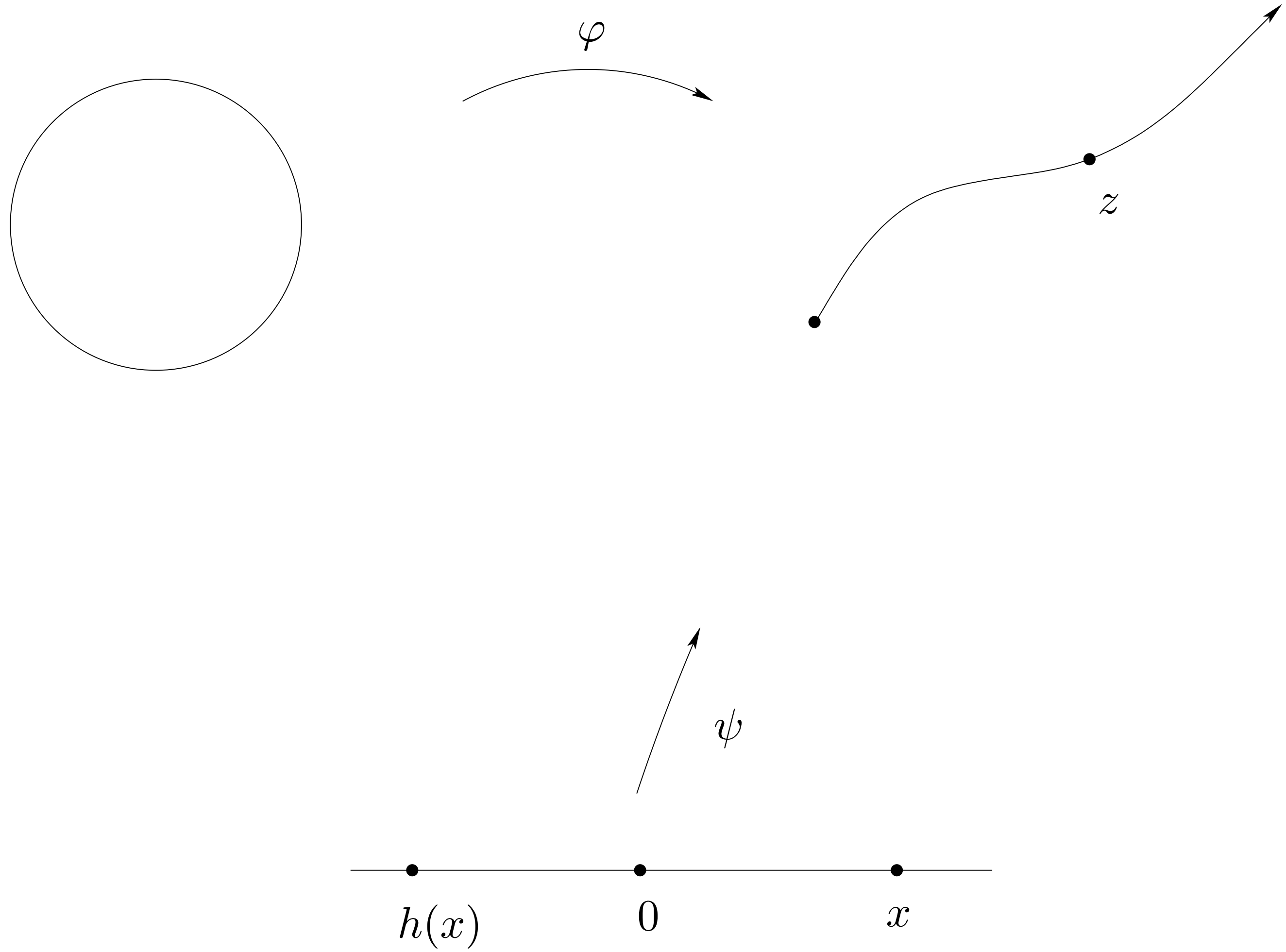
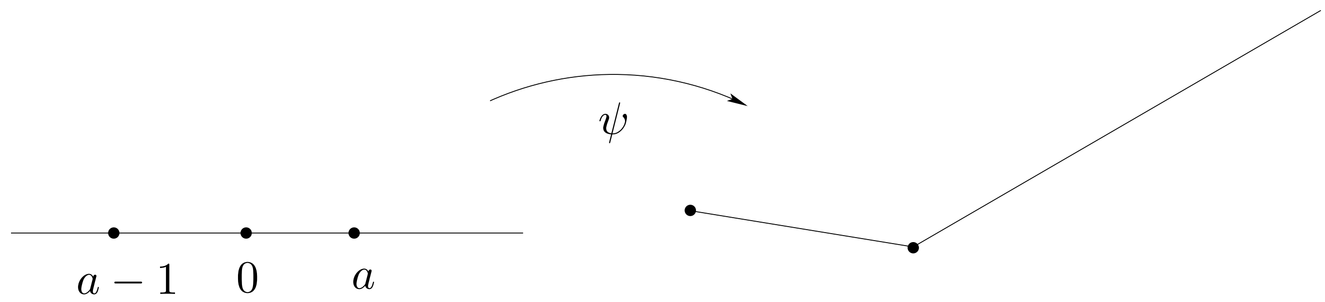
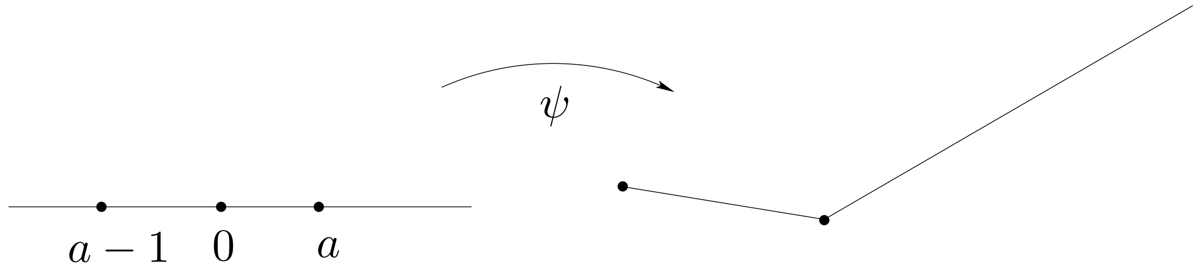
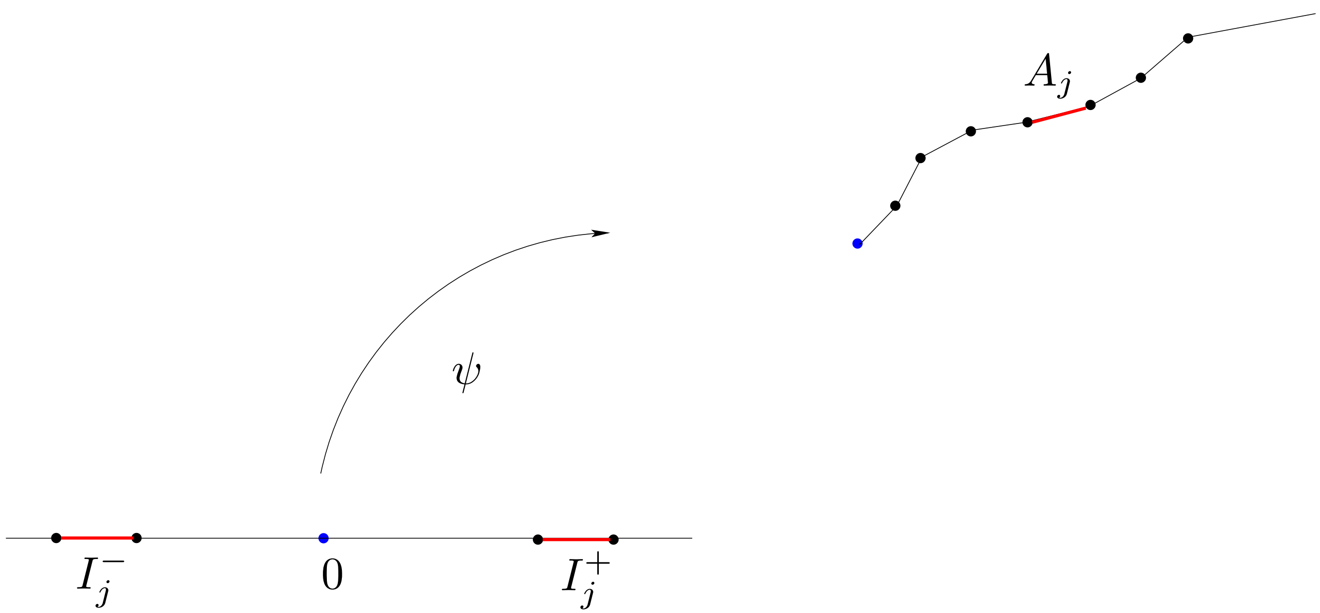


$$z = \psi(x) = \psi(h(x))$$

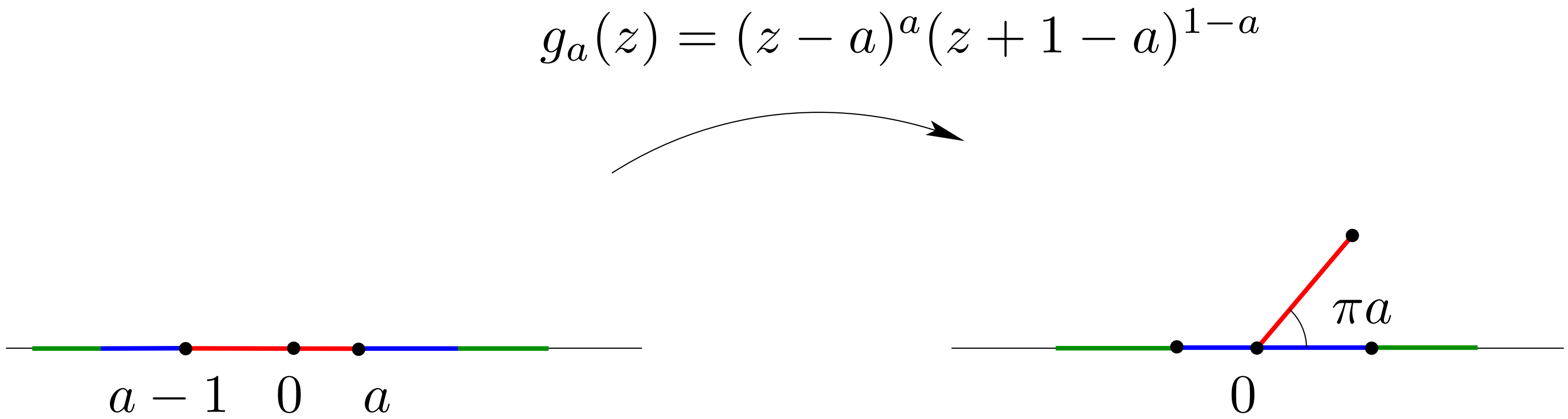

ψ
$a-1$
0
a

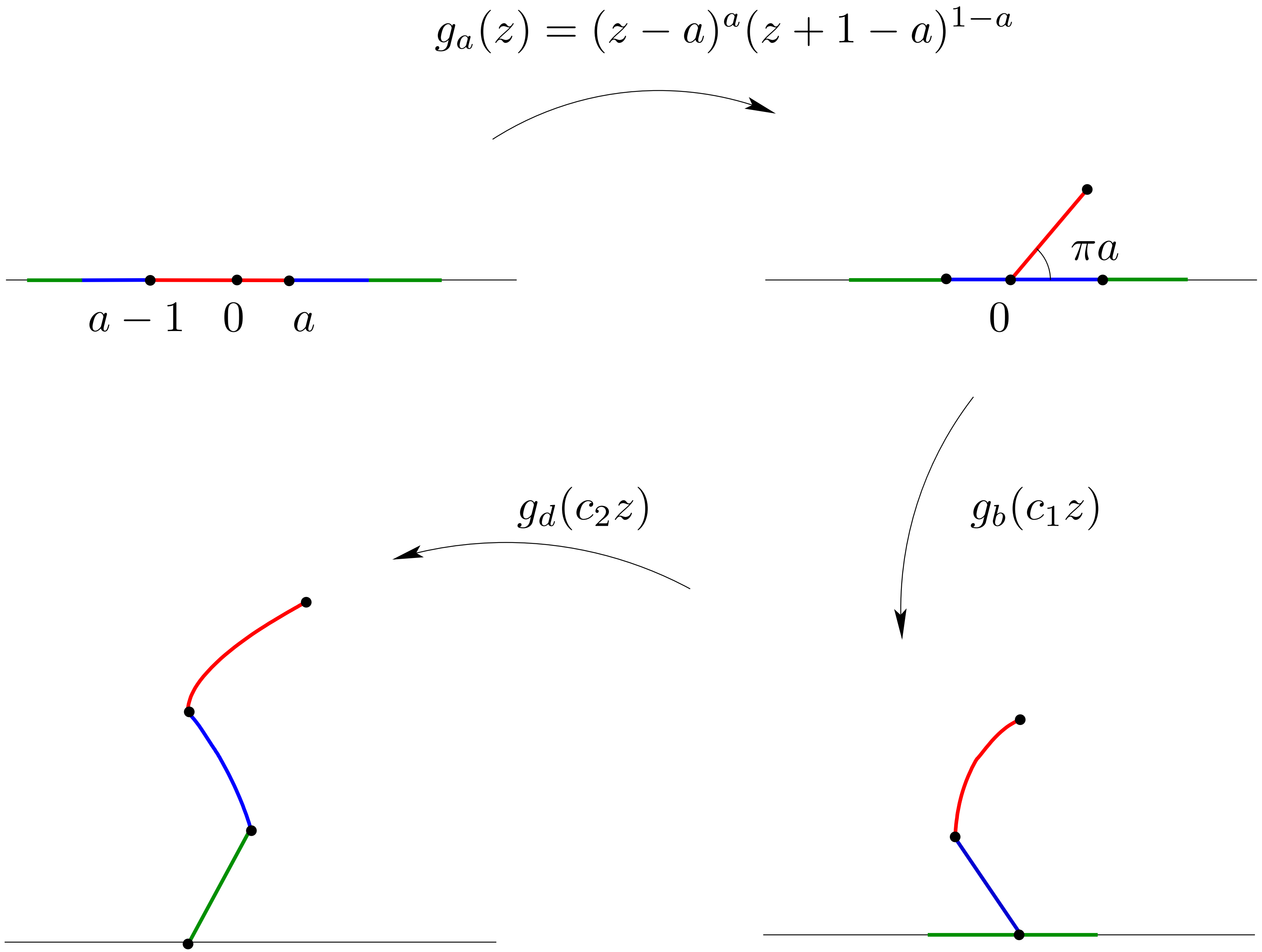


ψ
$a-1$
0
a



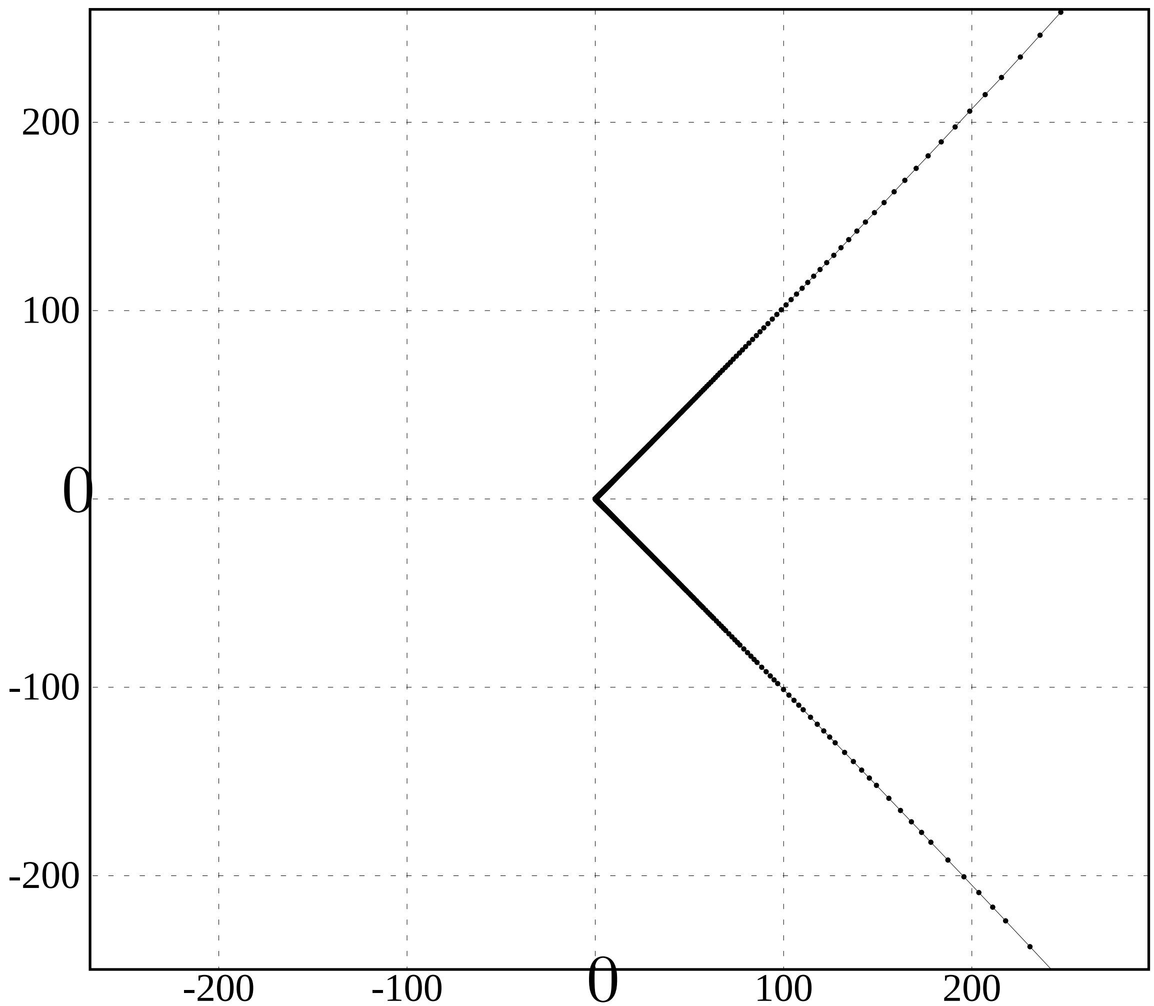
A_j
ψ
I_j^-
0
I_j^+

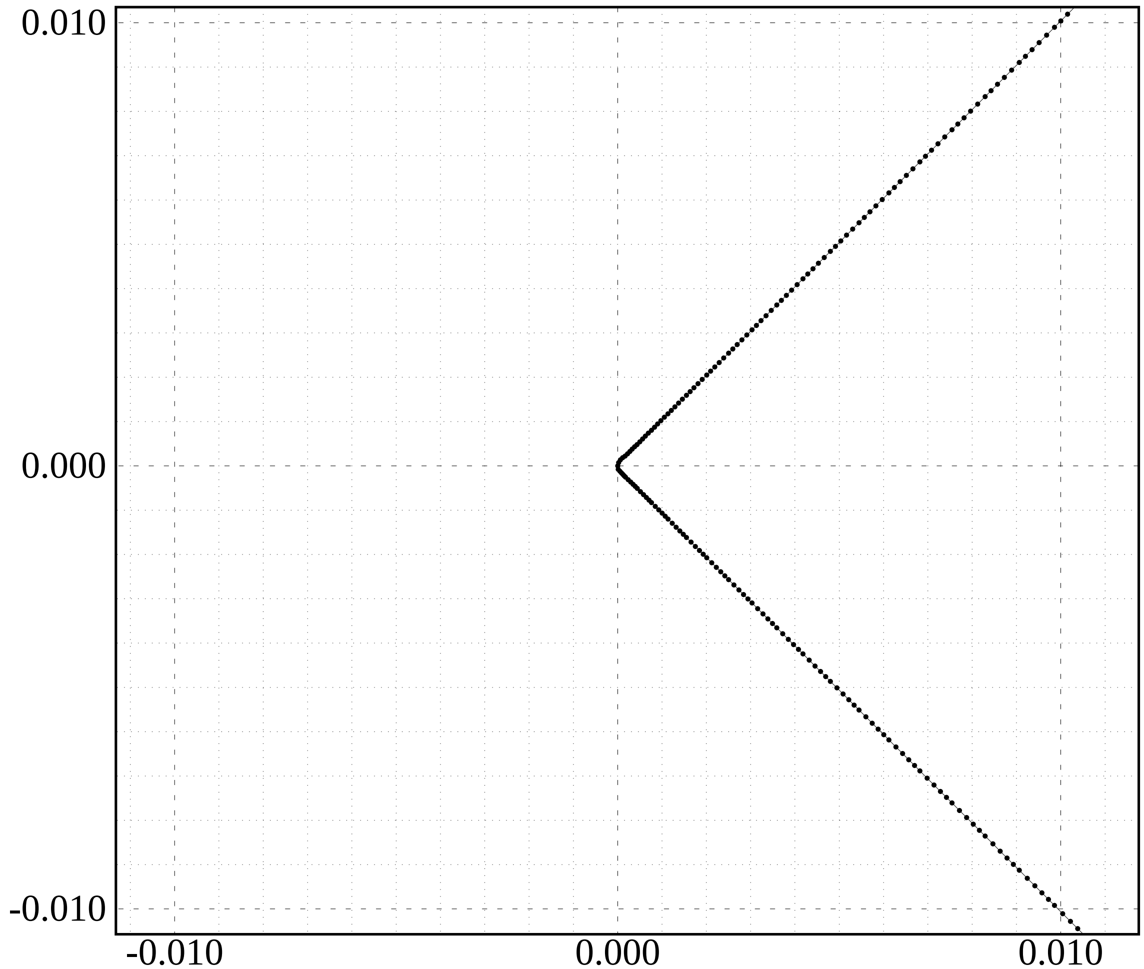

$g_a(z) = (z-a)^a(z+1-a)^{1-a}$
$a-1$
0
a
πa
0

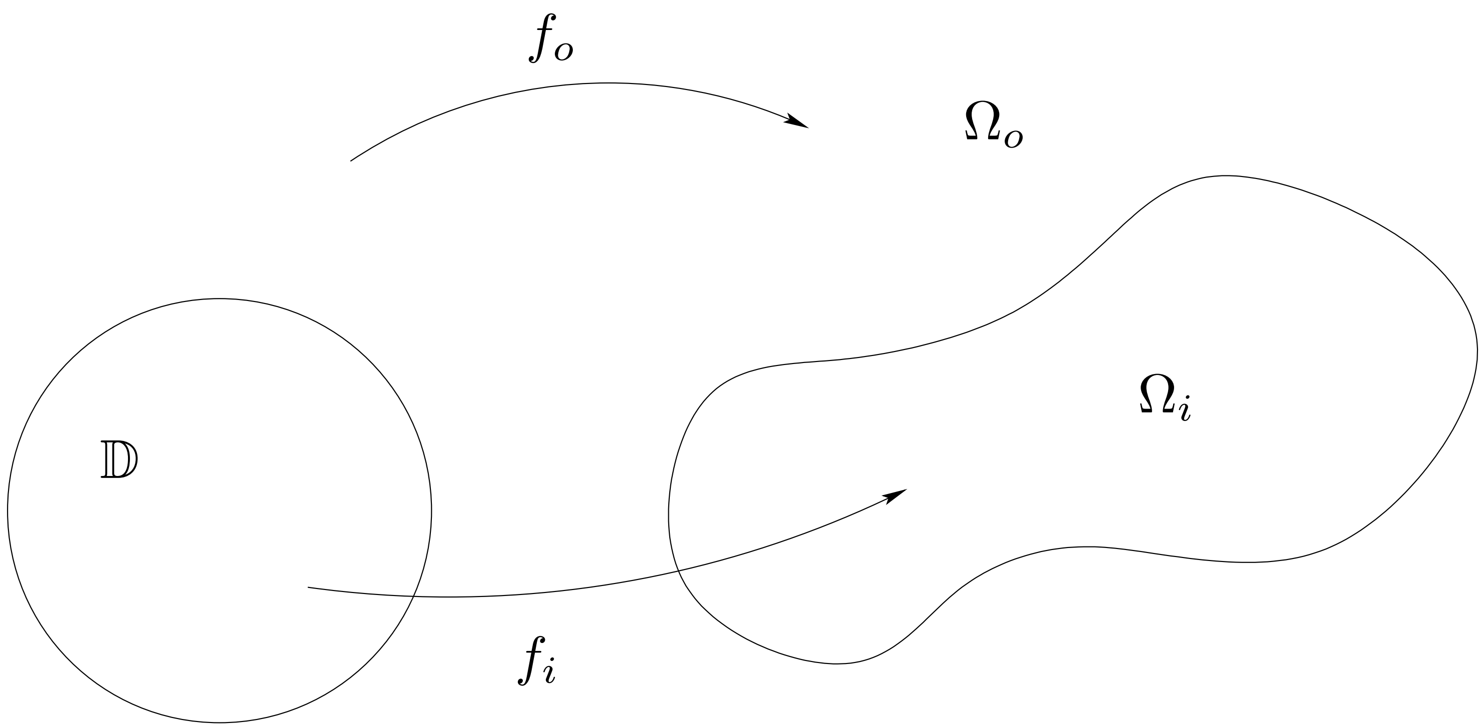


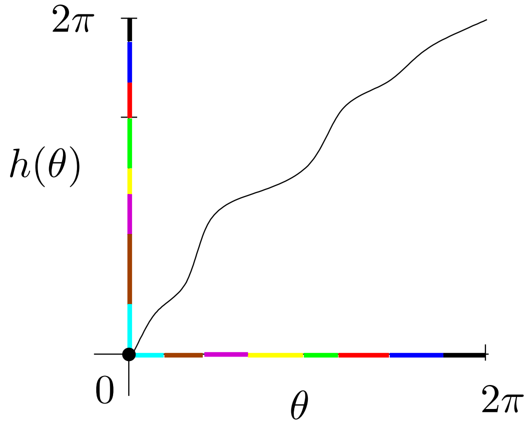
$g_a(z) = (z-a)^a(z+1-a)^{1-a}$
$a-1$
0
a
πa
0
$g_b(c_1 z)$
$g_d(c_2 z)$

$$h(x) = x^{\frac{1}{3}} \quad \text{on } \mathbb{R}$$




0.010
0.000
-0.010
-0.010
0.000
0.010



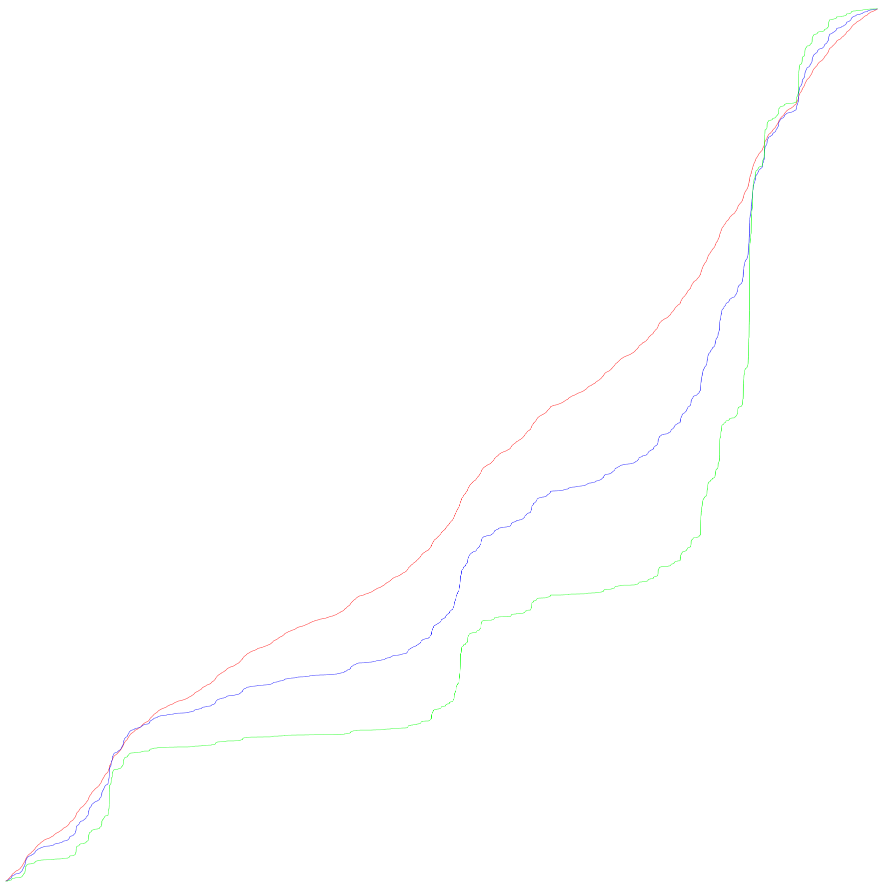


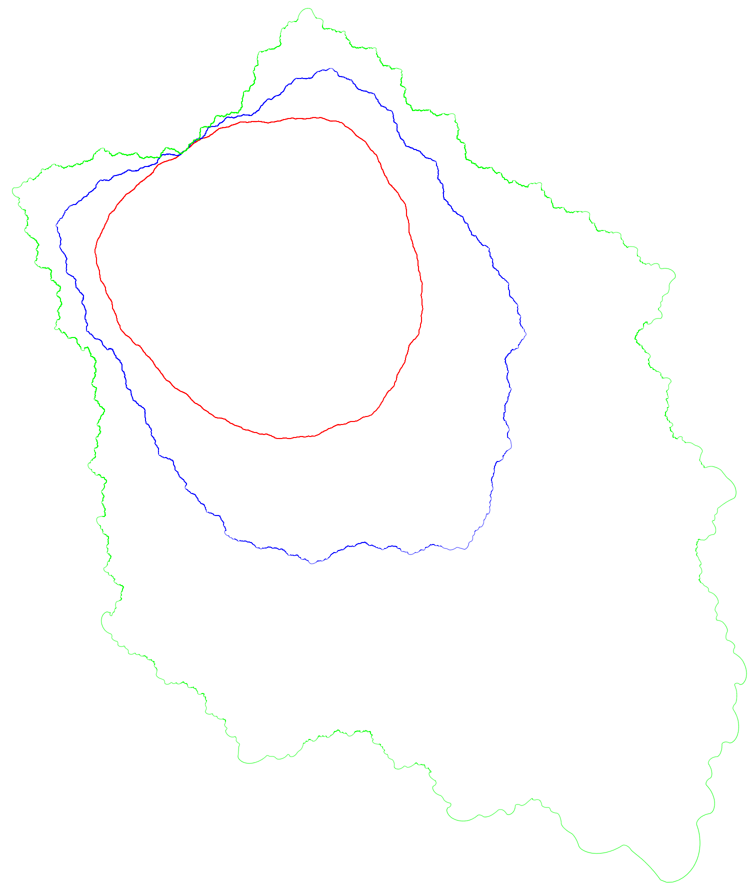
$$e^{ih(\theta)} = f_i^{-1} \circ f_o(e^{i\theta})$$

$\Gamma \in QC \iff h \in QS$:

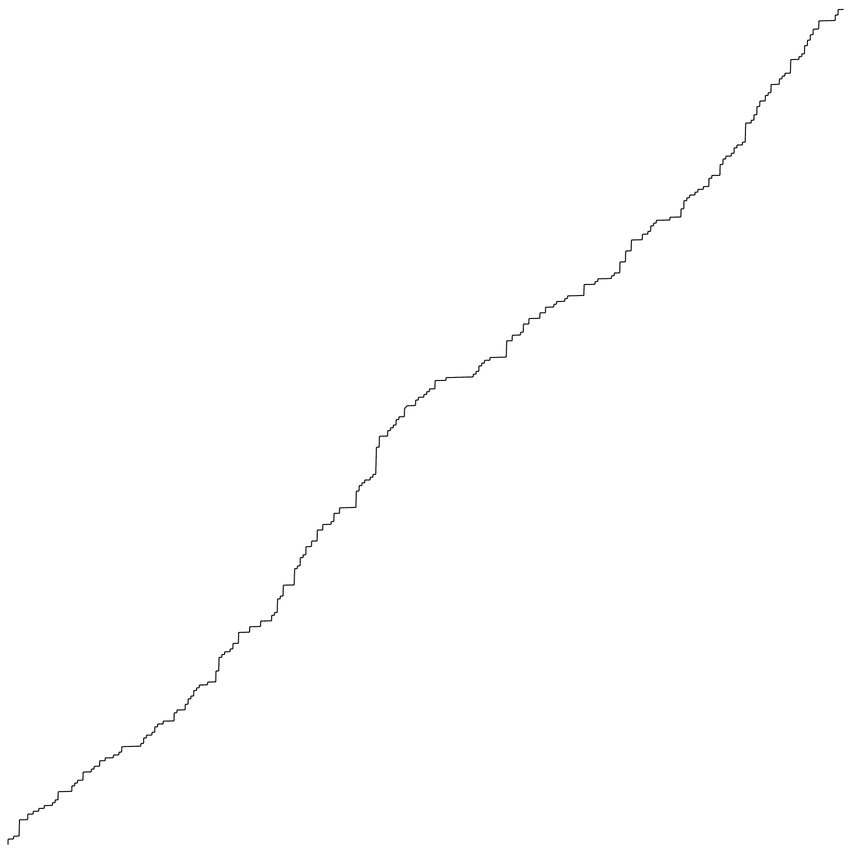
$|I| = |J|$, I, J adjacent $\Rightarrow |h(I)| \sim |h(J)|$.

Random dyadic QS welding

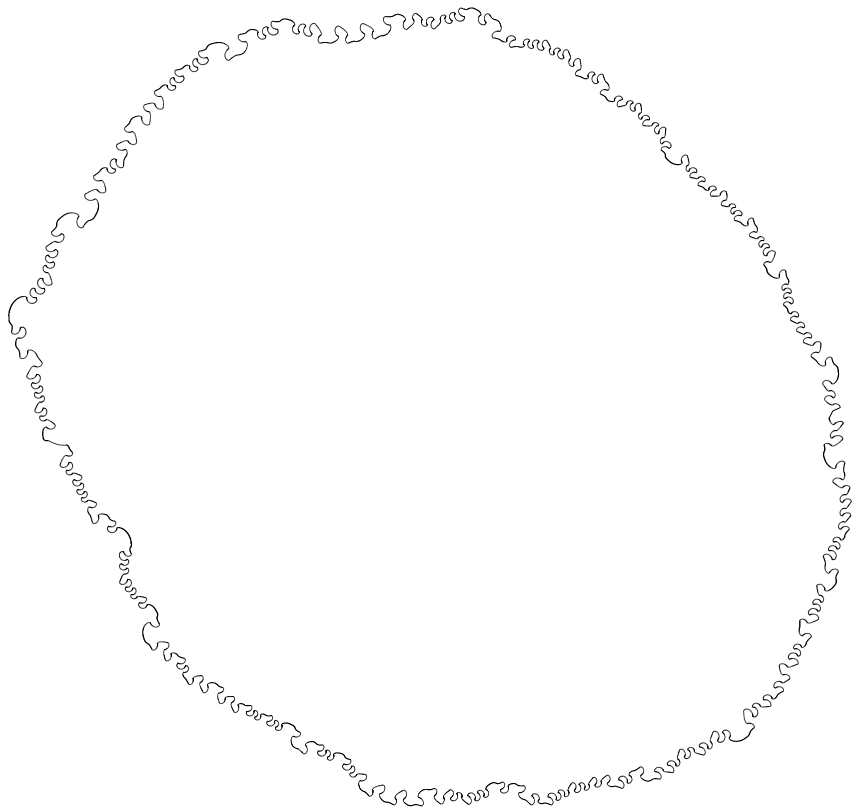


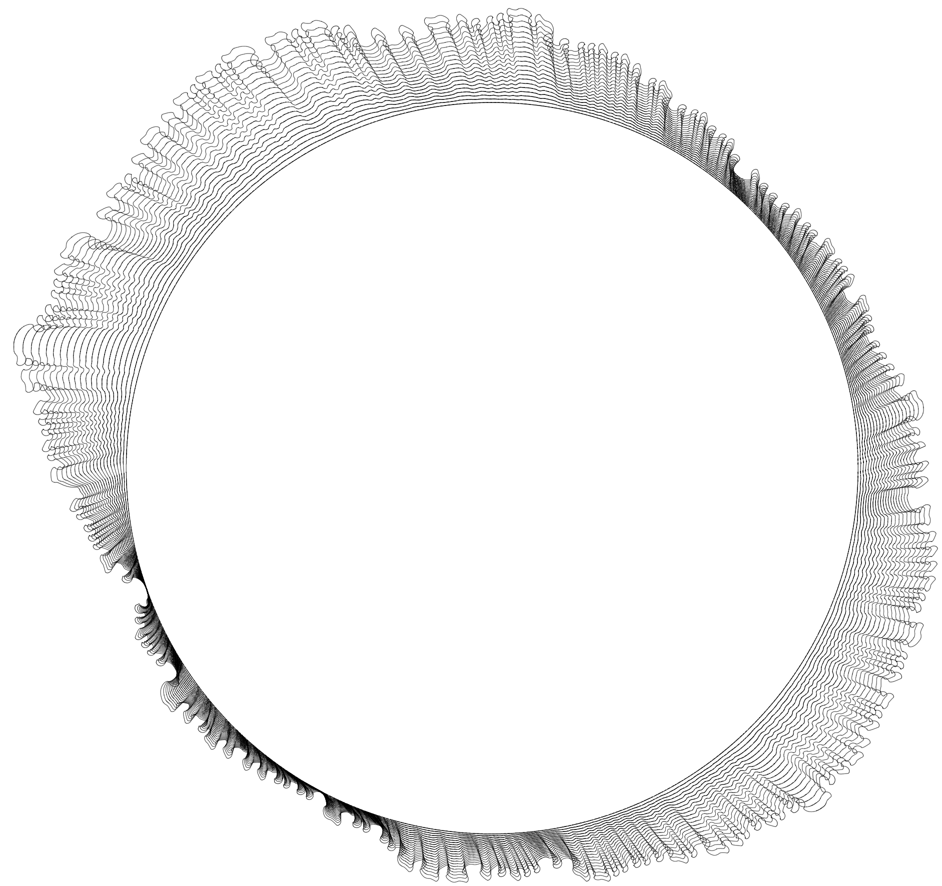


Random h: 300 steps with slopes $\sim 0, \infty$



Random image curve, 100 points on each segment (30,000 total)



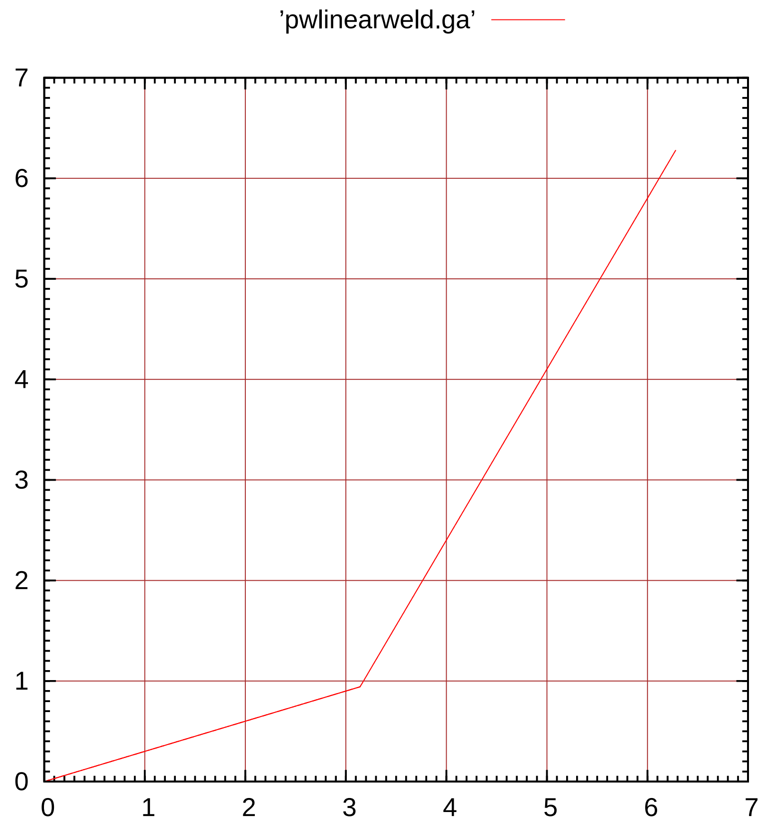


$$\text{smooth } \Gamma,\ z_0 \quad \overset{1-1}{\longleftrightarrow} \quad \text{smooth } h,\ h^{-1}$$

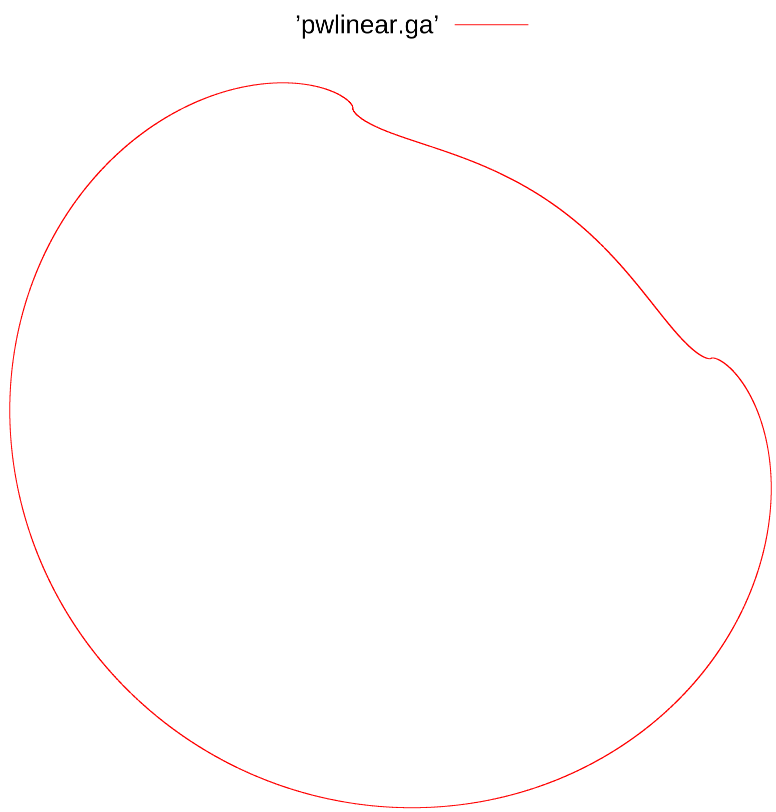
Problems:

$$\exists\ \Gamma_1, \Gamma_2 \quad \longrightarrow \quad h$$

$$\exists\ h \quad \not\longrightarrow \quad \Gamma$$

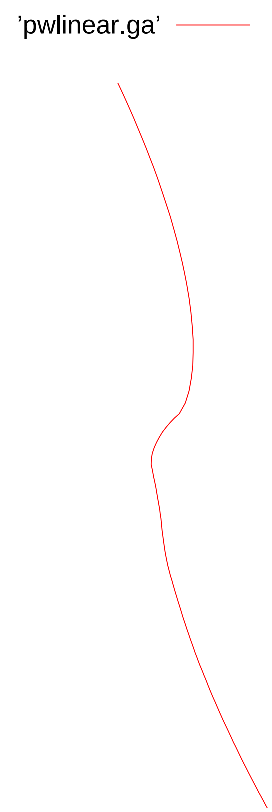


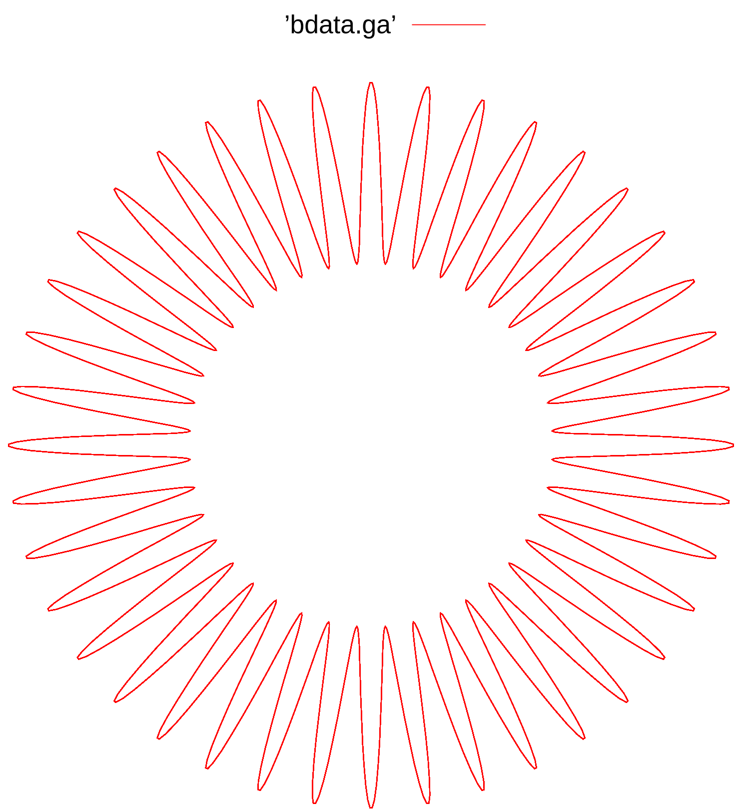
'pwlinearweld.ga'
0
1
2
3
4
5
6
7

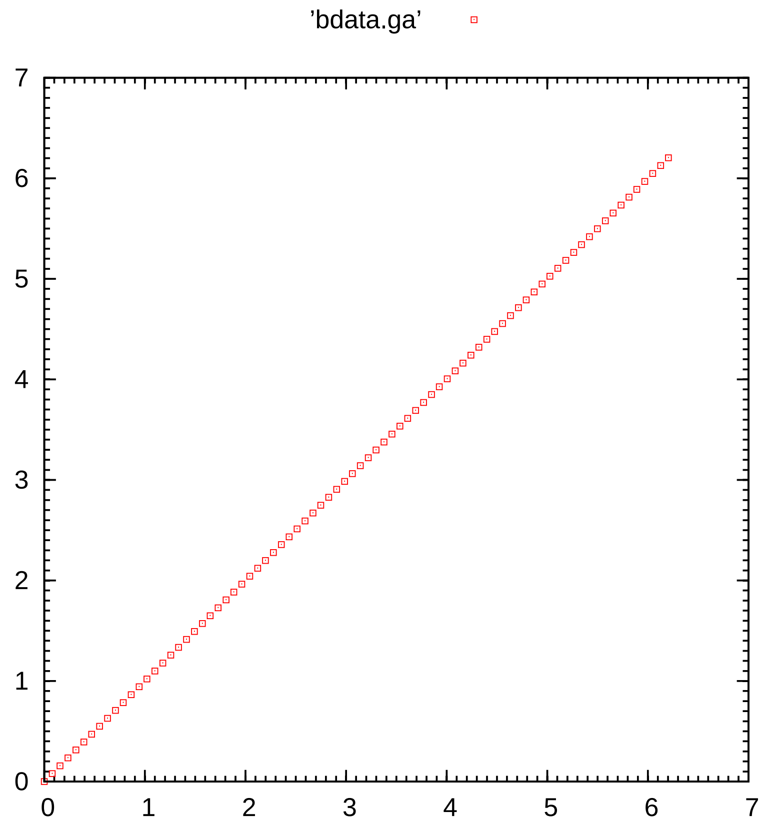

'pwlinear.ga'


'pwlinear.ga'


'pwlinear.ga'


'pwlinear.ga'


'bdata.ga'


'bdata.ga'
0
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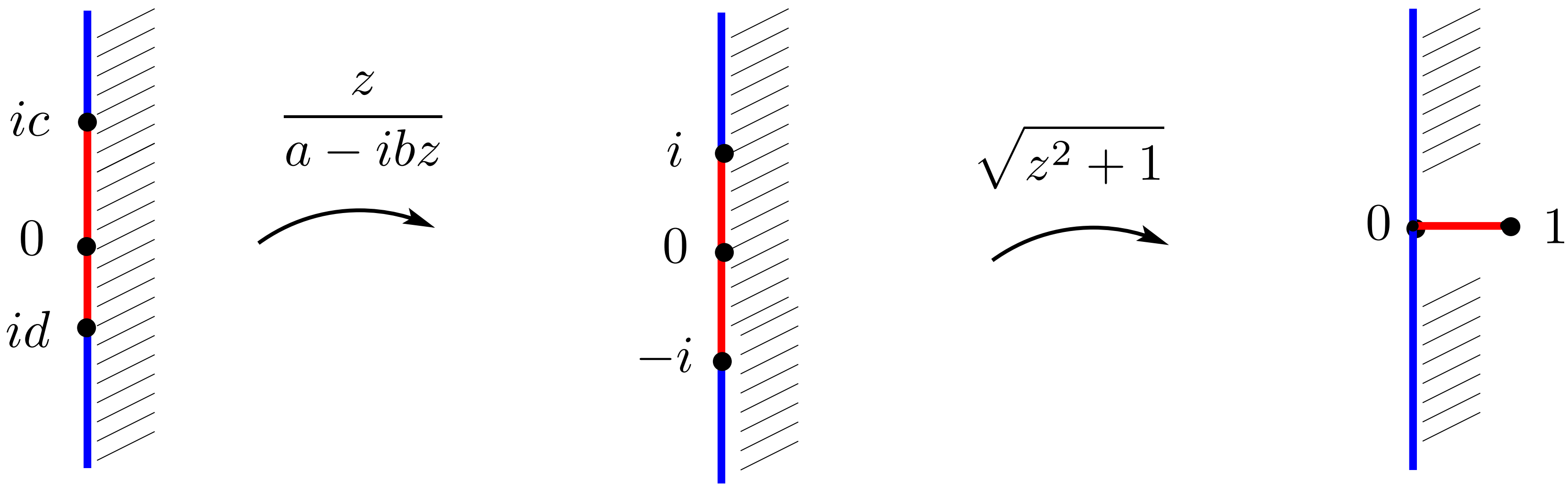
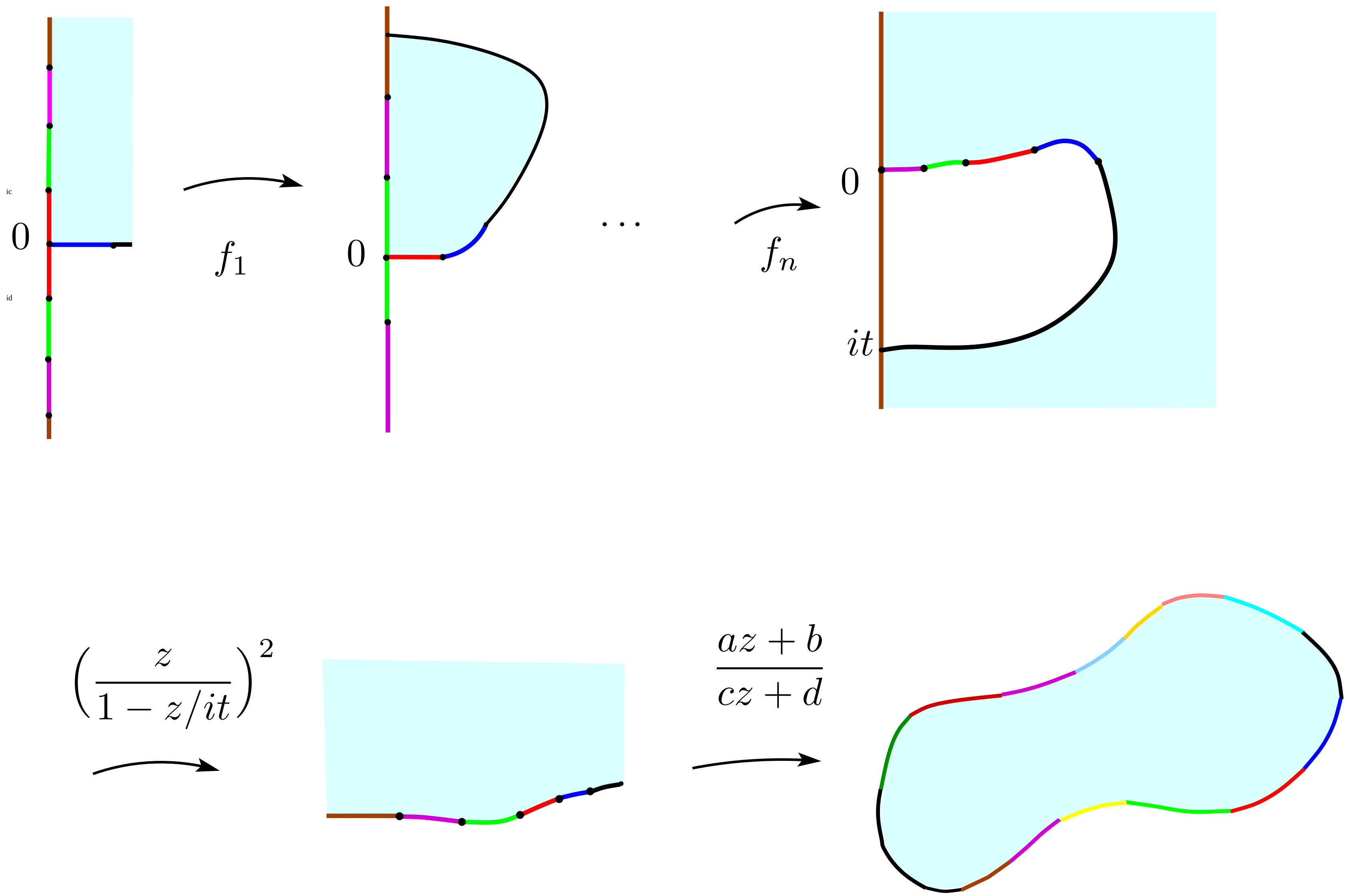
Basic Maps: $a > 0$, $b \in \mathbb{R}$





ic
0
id
f_1
0
$\cdots$
f_n
0
it
$\left(\frac{z}{1-z/it}\right)^2$
$\frac{az+b}{cz+d}$

Theorem 1 (Marshall-Rohde 2006).

$$\{z_j\}_1^n \in \Gamma \in C^1 \quad \textit{with } |z_j - z_{j+1}| \leq \mu_n,$$

then the Geodesic Algorithm produces conformal maps $f_i^{(n)}$ *and* $f_o^{(n)}$ *and welding maps* $h^{(n)}$ *which converge uniformly to the conformal maps and welding map for* Γ *as the mesh size* $\mu_n \to 0$. *The unit tangent vectors to the approximating curves also converge to the tangent directions of* Γ.

Conversely given a welding map h, the algorithm produces conformal maps $f_i^{(n)}$ and $f_o^{(n)}$ and welding maps $h^{(n)}$ such that $h^{(n)}$ converge uniformly to h.

Can exhaust an arbitrary s. c. domain by C^1 Jordan domains, and thereby give a constructive proof of the Riemann mapping Theorem.

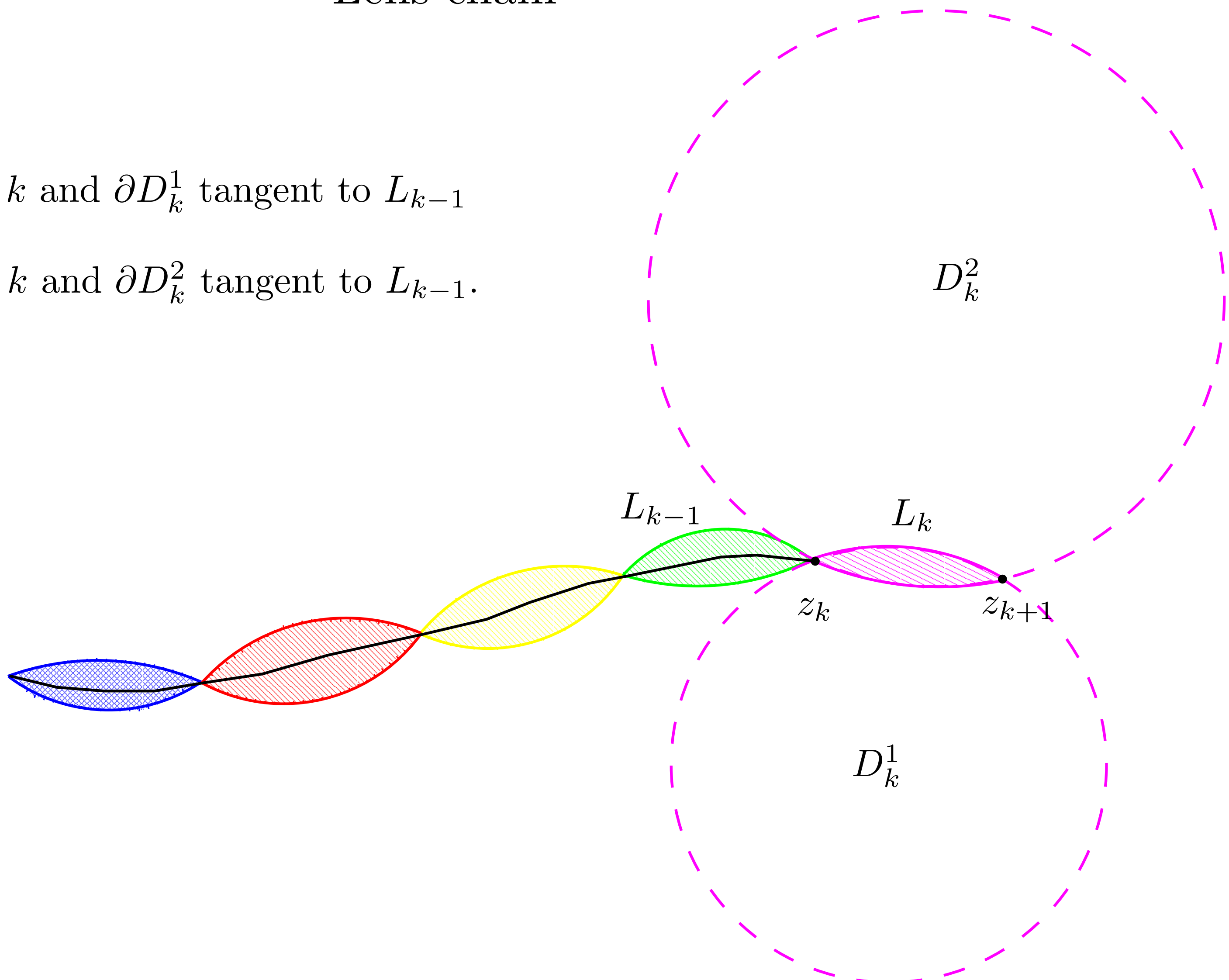
Theorem 2 (Marshall-Rohde 2007). *If Γ is a $K-$quasicircle, with $K < K_0$ then we can choose data points $\{z_j\}_1^n$ so that the geodesic algorithm finds a conformal map of $\mathbb{H}$ onto a region Ω_c bounded by a $C(K)-$quasicircle containing the data points, where $C(K)$ is a constant depending only on K.*

Lens-chain

$L_k = D_k^1 \cap D_k^2$

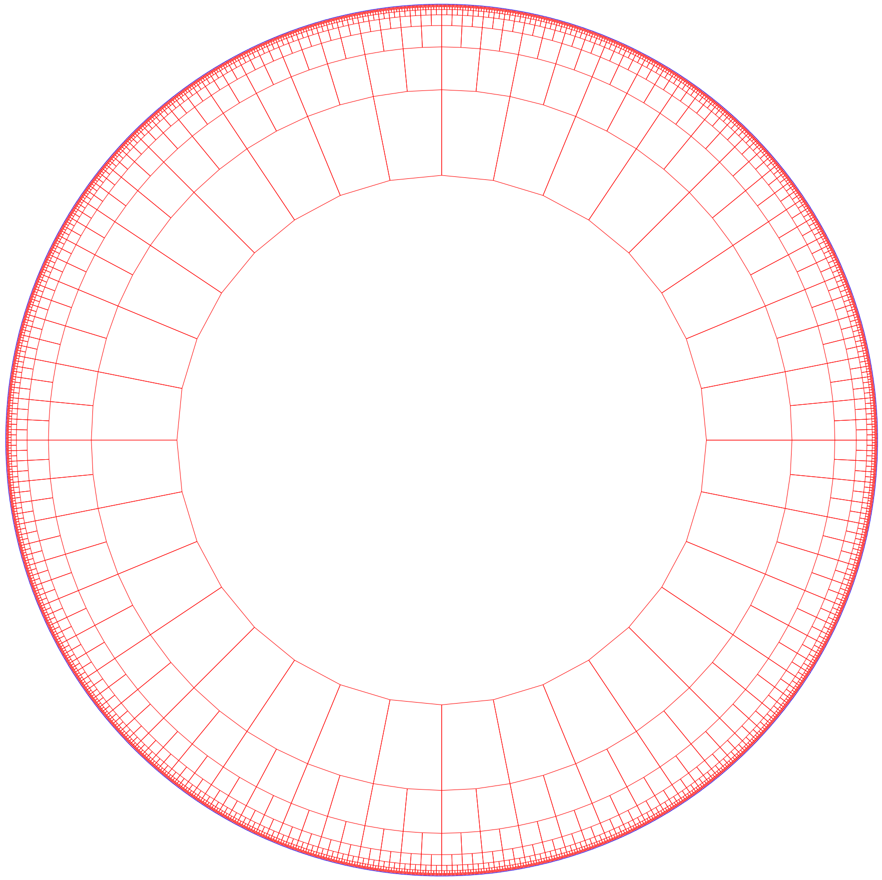
$D_k^1 \cap L_j = \emptyset$ for $j < k$ and ∂D_k^1 tangent to L_{k-1}

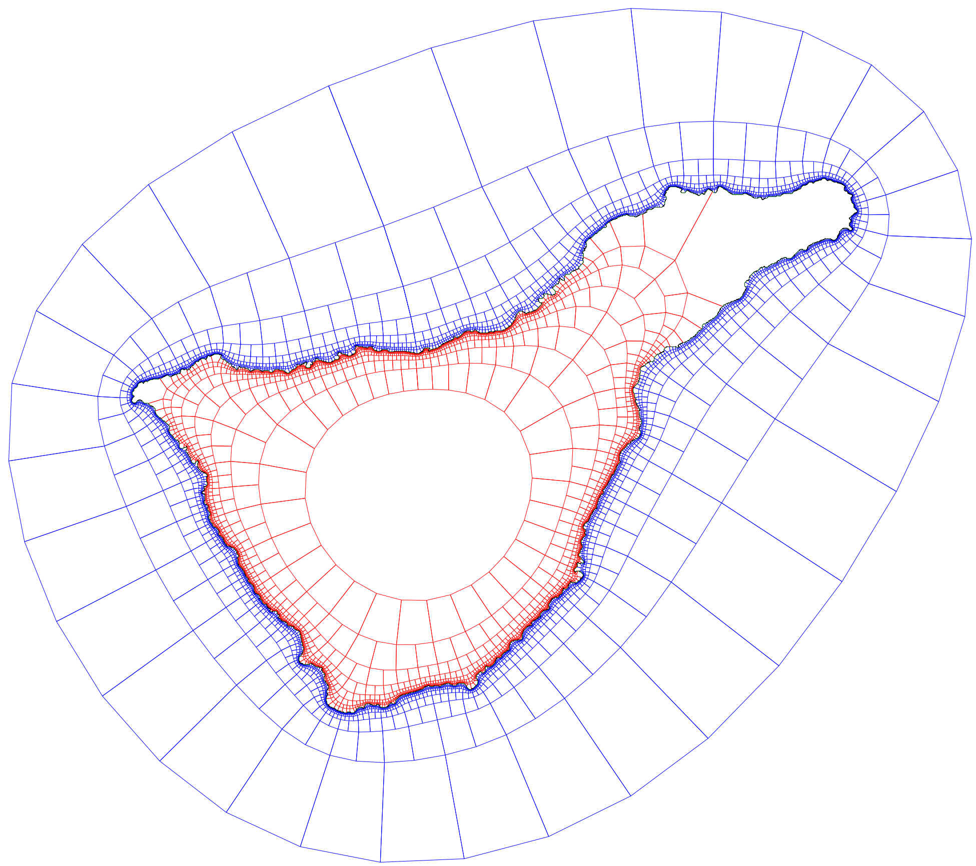
$D_k^2 \cap L_j = \emptyset$ for $j < k$ and ∂D_k^2 tangent to L_{k-1}.

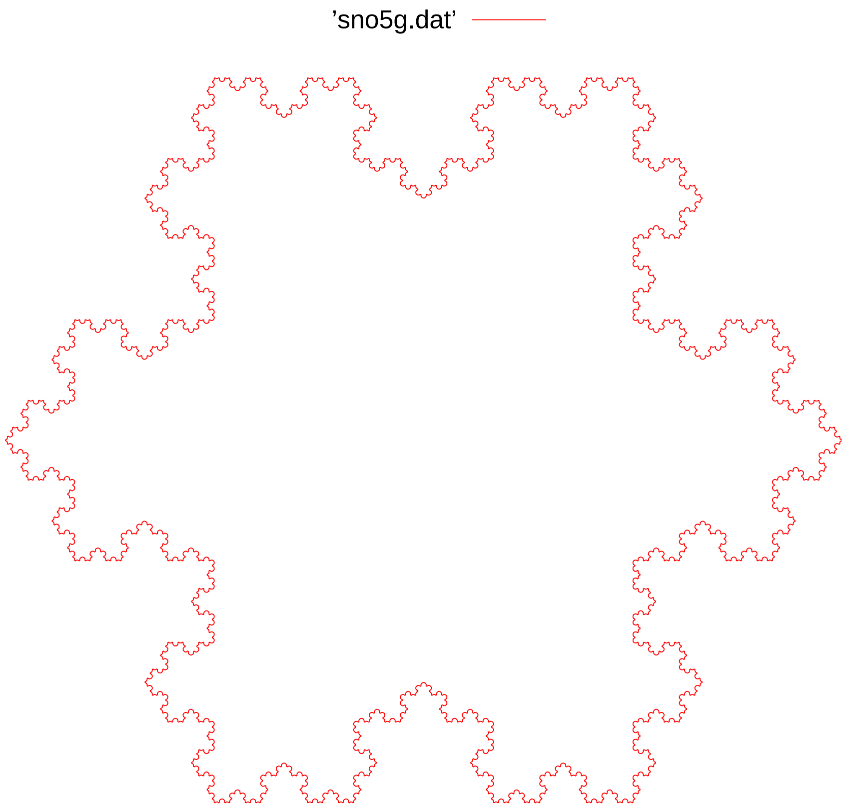


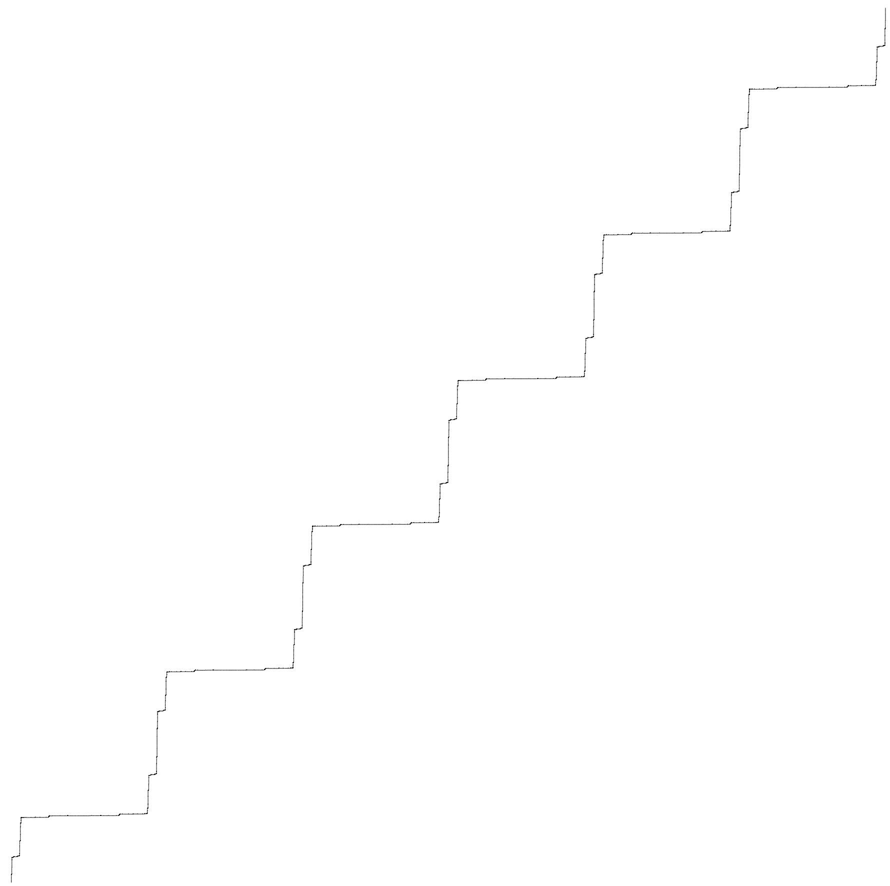
- simple
- fast and accurate: 10,000 boundary points takes 25 seconds on a 3.2Ghz Pentium 4 PC: $O(n^2)$ to compute the map, $O(n)$ to evaluate at a point.
- always works (need only $z_k \notin (z_{j-1}, z_j)$, for $k > j$)
- computes both φ and φ^{-1} for the region and its complement simultaneously
- composed map is always conformal
- uniform estimates (with proof).
- zipper version handles corners

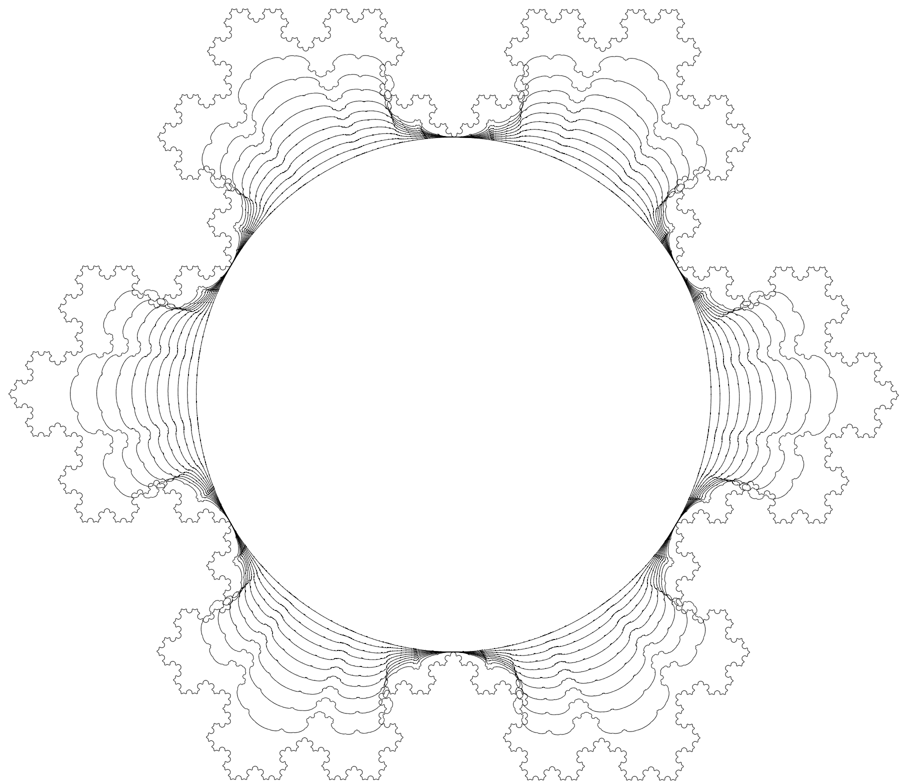
The algorithm is well-suited to use Driscoll and Vavasis's beautiful idea of using cross ratios to handle compression and expansion of conformal maps.

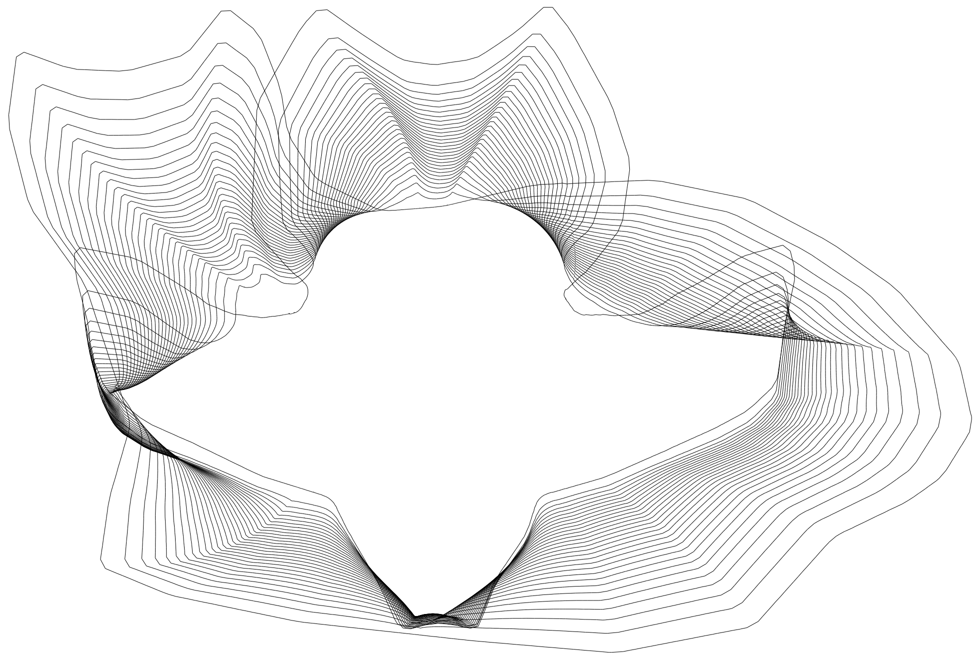


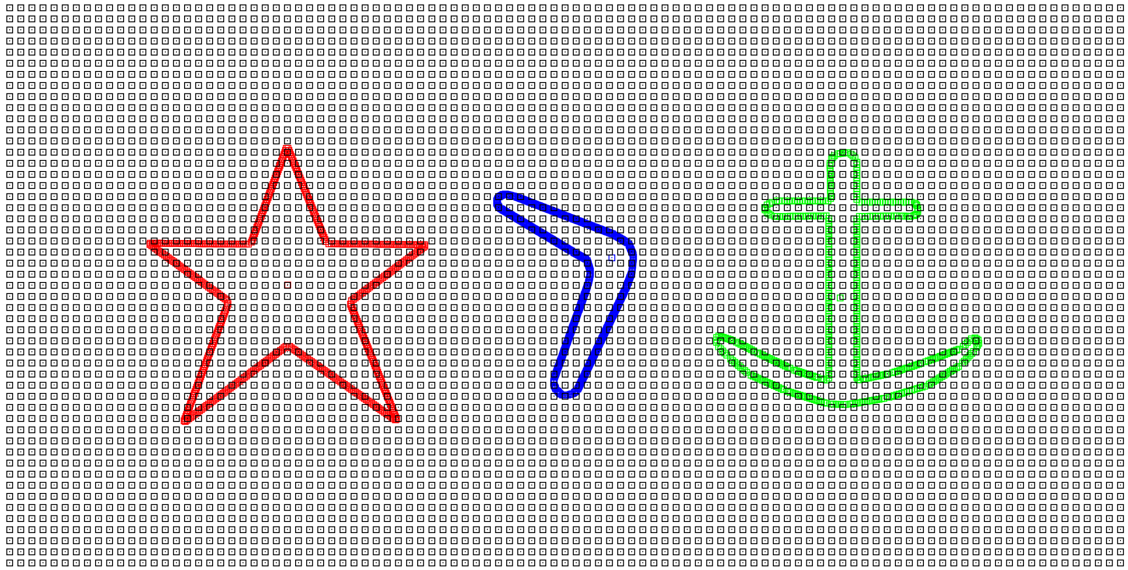


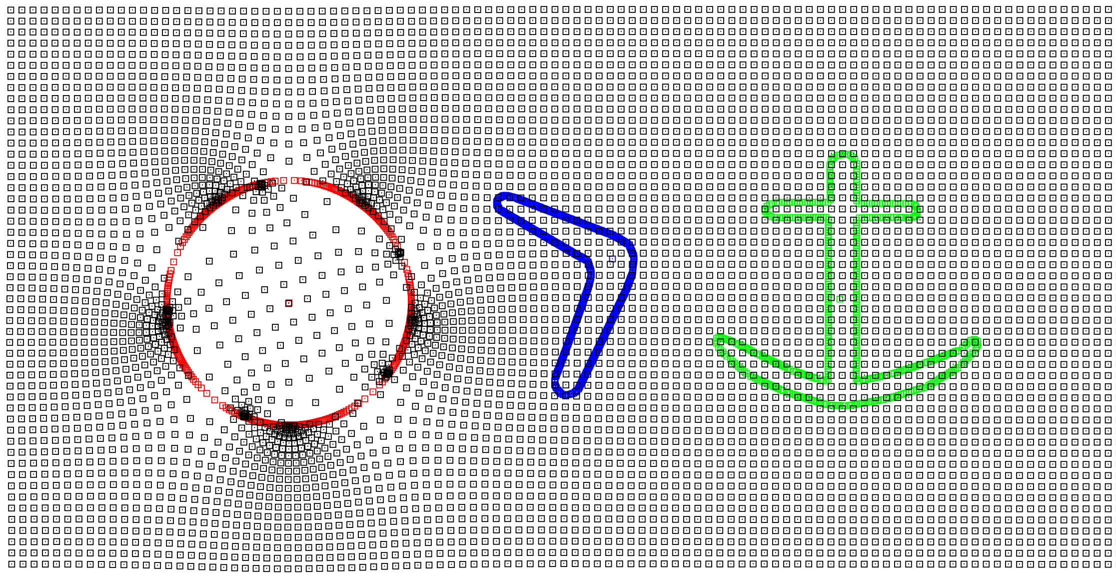

'sno5g.dat'

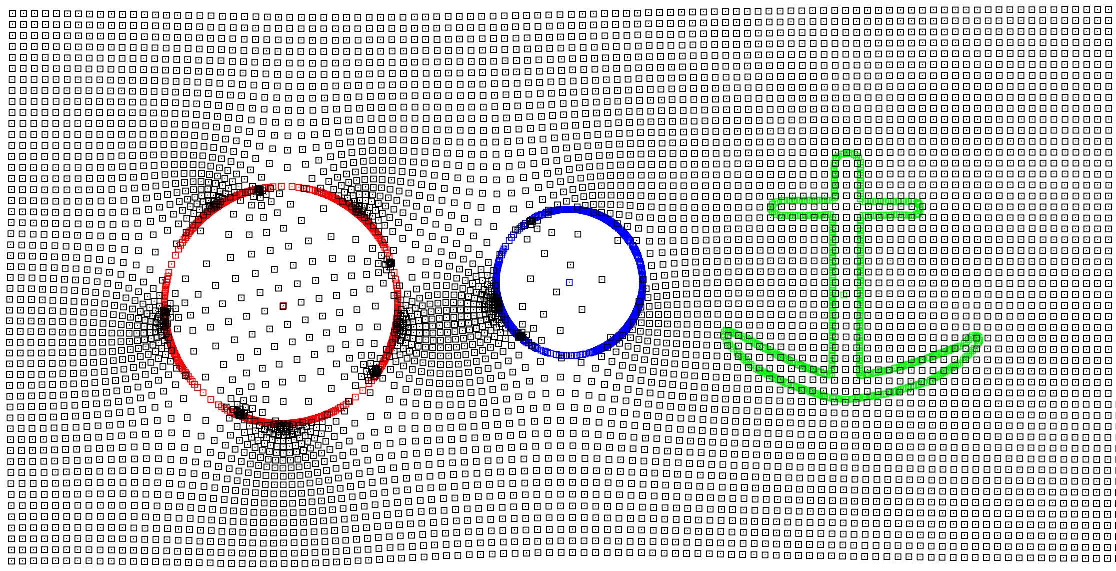


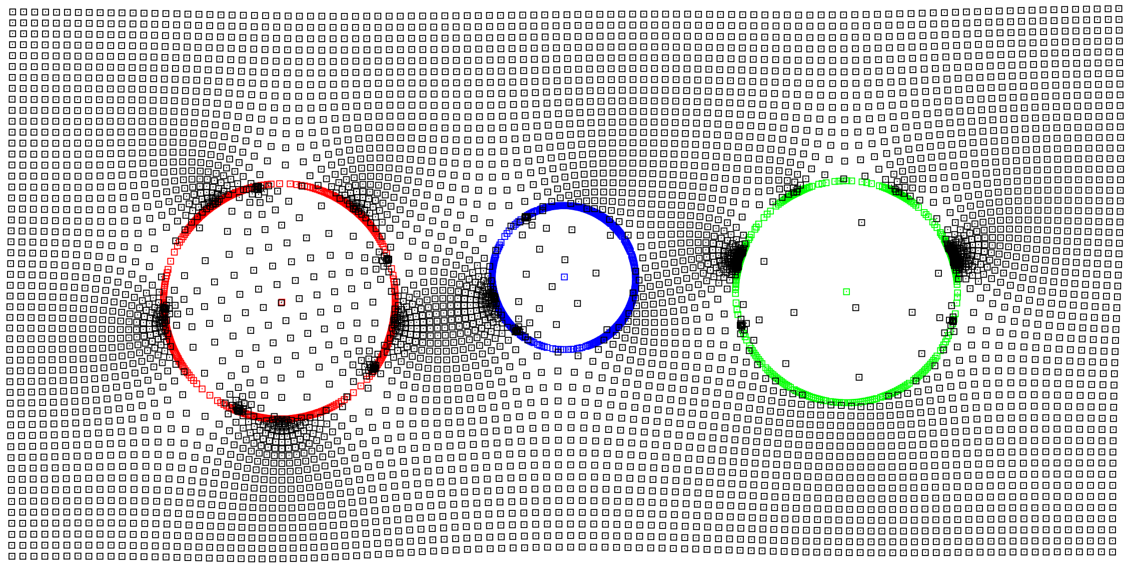












Open: how to apply multipole ideas to this algorithm

Tom Kennedy reduced O(n) to O($\mathrm{n}^{0.4}$) experimentally for computing SLE traces.

Open: If $h \in QS$ then is $h_n \in QS$ with comparable norm?

If so then the corresponding curves converge in the Hausdorff metric.

Open: Prove convergence for the zipper algorithm: angled slits instead of geodesics.