Conformal Geometry Applied in Computer Science

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Computational and Conformal Geometry

Collaborators

The work is collaborated with the

Mathematicians

Shing-Tung Yau, Feng Luo, Zeng-Xue He

Computer Scientists

Arie Kaufman, Hong Qin, Dimitris Samaras, Klaus Mueller, Joe Mitchell, Esther Arkin, Jie Gao

Artist

Lance Cong

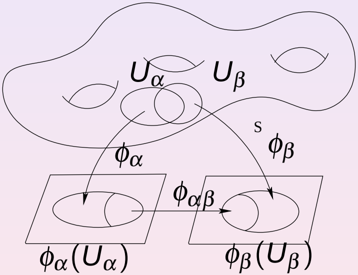
and many faculty members in computer science department in Stony Brook University.

Collaborators

The work is implemented by many students in the *Center of Visual Computing*. Especially, Miao Jin, Junho Kim, Xiaotian Yin, Wei Zeng and Xin Li.

Definition (Conformal Structure)

An atlas is conformal, if all its transition maps are conformal (biholomorphic). A conformal structure is the maximal conformal atlas. A topological surface with an conformal structure is called a Riemann Surface.



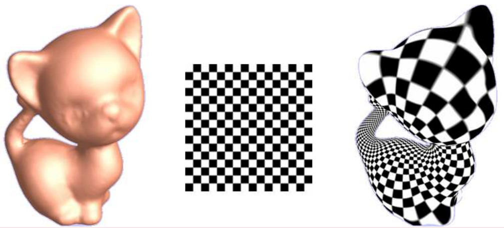
Relation between conformal structure and Riemannian metric

Isothermal Coordinates

A surface Σ with a Riemannian metric $\mathbf{g}$, a local coordinate system (u,v) is an isothermal coordinate system, if

$$\mathbf{g} = e^{2u}(du^2 + dv^2).$$

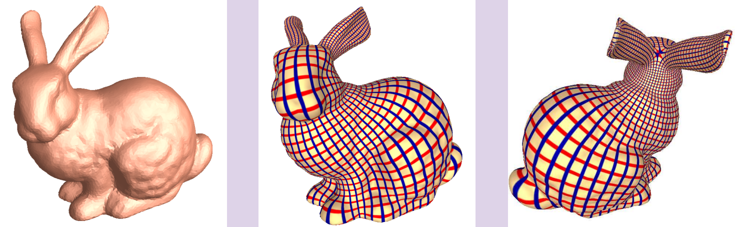
The atlas formed by isothermal coordinate systems is an conformal atlas.



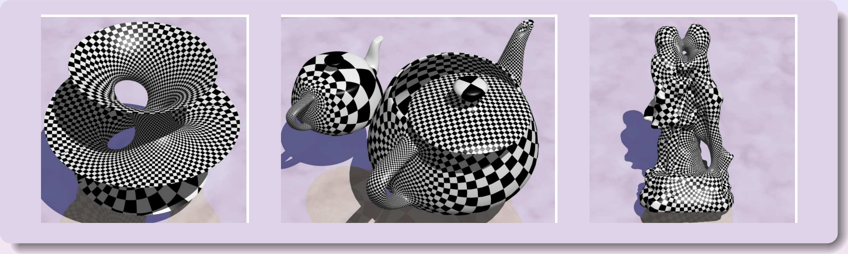
Riemann Surface

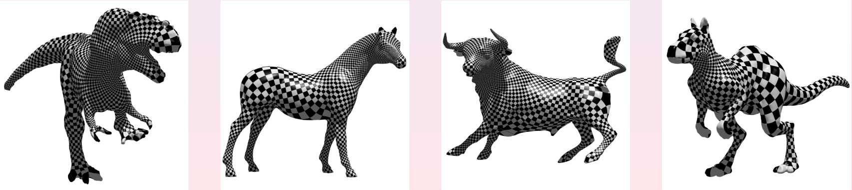
All metric surfaces are Riemann surfaces.

Conformal Structure



Conformal Structure





Computational Method

Heat Flow

Suppose the temperature field on the surface is $T(u,v,t)$, the surface is with a Riemannian metric $\mathbf{g}$, then the temperature will evolve according to the heat flow:

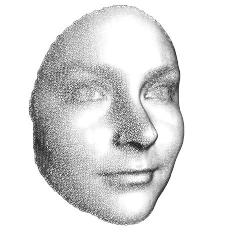
$$\frac{dT(u,v,t)}{dt} = \Delta_{\mathbf{g}} T(u,v,t),$$

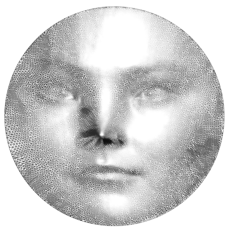
at the steady state

$$\Delta_{\mathbf{g}} T(u,v,\infty) \equiv 0,$$

which is called a harmonic function.

Heat Flow Acting on Linear Maps



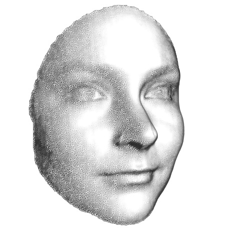


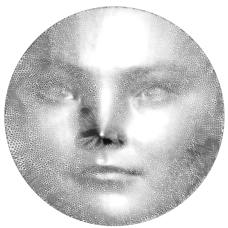
Linear Harmonic Maps

Heat flow acting on the maps

$$\frac{d\phi(u,v,t)}{dt} = \Delta\phi(u,v,t).$$

Heat Flow Acting on Linear Maps





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$$\frac{d\phi(u,v,t)}{dt} = \Delta\phi(u,v,t).$$

Heat Flow Acting on nonlinear Maps

Non-linear Harmonic Maps

Heat flow acting on the maps

$$\frac{d\phi(u,v,t)}{dt} = \Delta\phi(u,v,t) - (\Delta\phi(u,v,t))^{\perp}$$

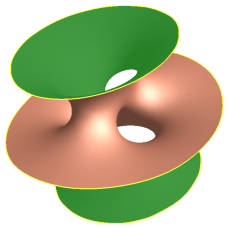
Heat Flow Acting on nonlinear Maps

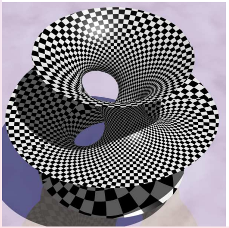
Non-linear Harmonic Maps

Heat flow acting on the maps

$$\frac{d\phi(u,v,t)}{dt} = \Delta\phi(u,v,t) - (\Delta\phi(u,v,t))^{\perp}$$

Heat Flow Acting on Vector Fields (Differential Forms)

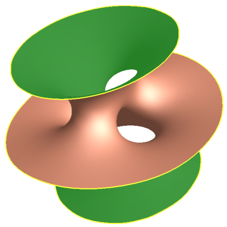


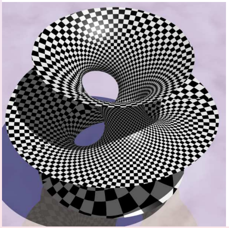


Holomorphic 1-forms

Heat flow acting on 1-forms, the heat flow is

$$\frac{d\omega(u,v,t)}{dt} = \Delta\omega(u,v,t).$$





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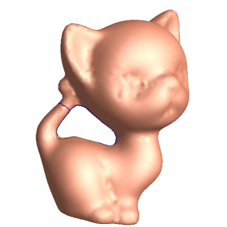
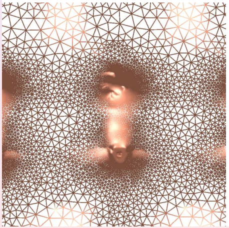
Heat Flow Acting on Metrics



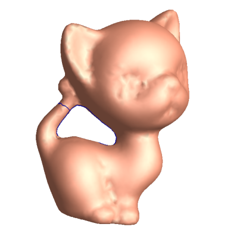


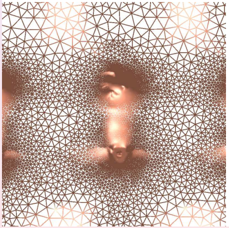
Euclidean Ricci Flow

Heat flow acting on metrics, the curvature satisfies the heat flow

$$\frac{dK(u,v,t)}{dt} = \Delta_{g(t)} K(u,v,t).$$

Heat Flow Acting on Metrics





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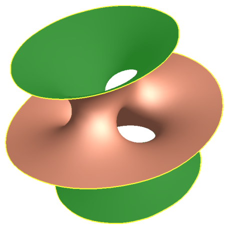

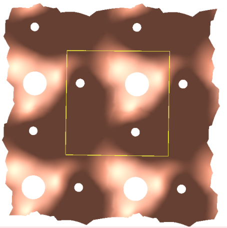

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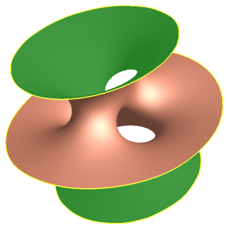

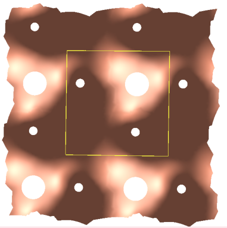

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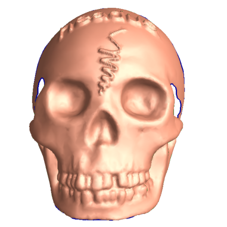

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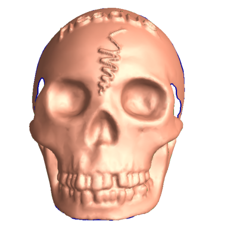

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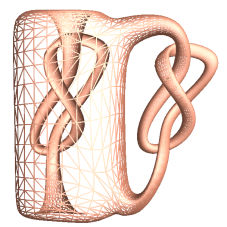

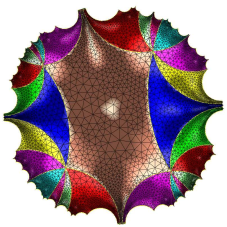

Hyperbolic Ricci Flow

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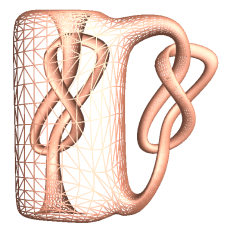

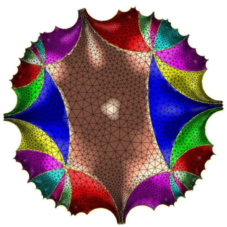

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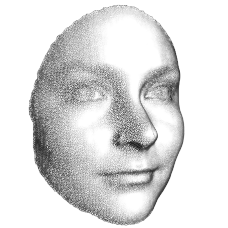
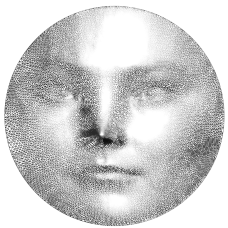
Heat Flow Acting on Linear Maps





Linear Harmonic Maps

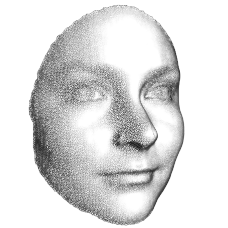
Heat flow acting on the maps

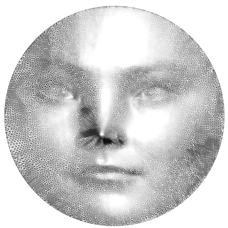
$$\frac{d\phi(u,v,t)}{dt} = \Delta\phi(u,v,t).$$

Theorem (Rado's theorem)

Assume $\Omega \subset \mathbb{R}^2$ is a convex domain with smooth boundary $\partial\Omega$. Given any homeomorphism $\phi : S^1 \to \partial\Omega$, there exists a unique harmonic map $u : D \to \Omega$, such that $u = \phi$ on $\partial D = S^1$ and u is a diffeomorphism.

Heat Flow Acting on Linear Maps





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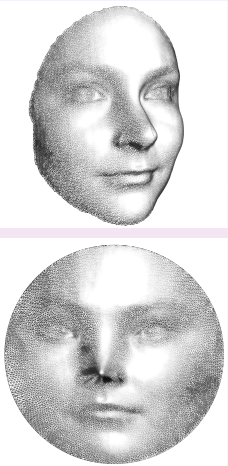


Finite Element Method

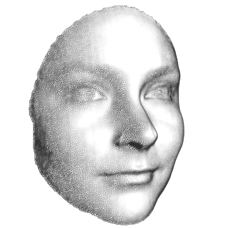
Given a mesh Σ, for an edge e_{ij} connecting vertices v_i and v_j, suppose two angles against e are α, β, the define *edge weight* as

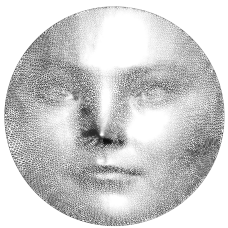
$$w_{ij} = \frac{1}{2}(\cot\alpha + \cot\beta)$$

suppose a map $\phi : \Sigma \to \mathbb{R}^2$, then the discrete energy is

$$E(\phi) = \sum_{e_{ij}} w_{ij} |\phi(v_i) - \phi(v_j)|^2.$$

Heat Flow Acting on Linear Maps





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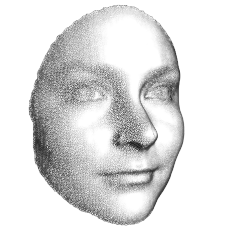
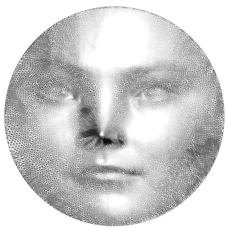




Finite Element Method

Discrete Laplace-Beltrami operator

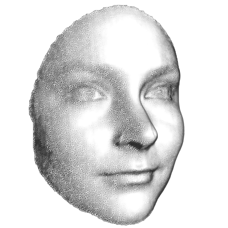
$$\Delta\phi(v_i) = \sum_{e_{ij}} w_{ij}(\phi(v_i) - \phi(v_j)),$$

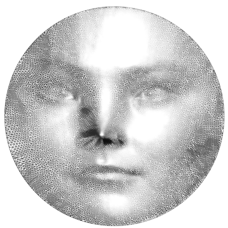
Heat flow

$$\phi(v_i) - = \Delta\phi(v_i)\varepsilon,$$

where ε is a small constant.

Heat Flow Acting on Linear Maps





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Spherical Conformal Maps

Non-linear Harmonic Maps

Heat flow acting on the maps

$$\frac{d\phi(u,v,t)}{dt} = \Delta\phi(u,v,t) - (\Delta\phi(u,v,t))^{\perp}$$

Theorem (Heat Flow for Topological Sphere)

The heat flow of a map from a closed genus zero surface to the unit sphere converges to a conformal map under normalization constraints. The conformal map is a diffeomorphism.

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Spherical Conformal Maps

Discrete Approximation

Heat flow acting on the maps

$$\phi(v_i) -= (\Delta\phi(v_i) - \Delta\phi(v_i)^{\perp})\varepsilon$$

where $\Delta\phi(v_i))^{\perp}$ is defined as

$$< \Delta\phi(v_i), \phi(v_i) > \phi(v_i).$$

Spherical Conformal Maps

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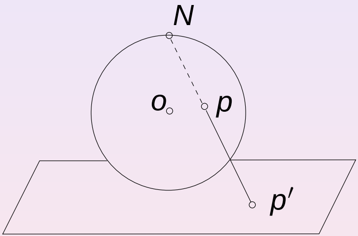
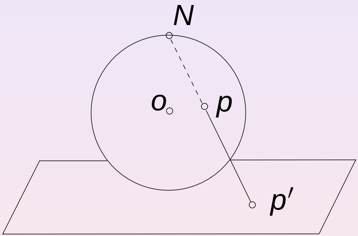


Stereo graphic projection

A conformal map from the unit sphere $p(x,y,z)$ to the complex plane

$$p' = \frac{2}{2-z}p,$$



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Spherical Conformal Maps

Möbius Transform

A Möbius transform on the complex plane $\phi : \mathbb{C} \to \mathbb{C}$ is

$$\phi(z) = \frac{az+b}{cz+d}, ad-bc = 1,$$

where $a, b, c, d \in \mathbb{C}$

Theorem (Conformal Automorphism Group)

The conformal maps from a unit sphere to itself (or the complex plane) differ by a Möbius map.

Normalization

In order to remove the Möbius ambiguity, spherical harmonic map in normalized

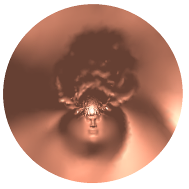
1. Compute the mass center of the image,

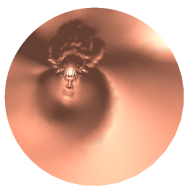
$$\mathbf{c} = \sum_{v_i} \phi(v_i),$$

2. Normalize

$$\phi(v_i) = \frac{\phi(v_i) - \mathbf{c}}{|\phi(v_i) - \mathbf{c}|}$$

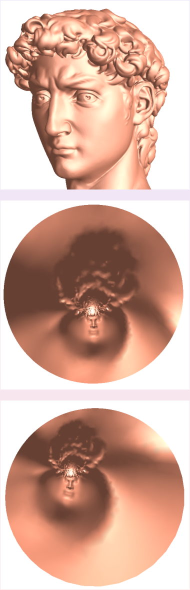
Riemann Mapping Theorem





Topological Disk Conformal Mapping

1. Double cover
2. Conformally map the doubled surface to the unit sphere
3. Use the sphere Möbius transformation to make the mapping symmetric.
4. Use stereographic projection to map each hemisphere to the unit disk.



Möbius Transformation

A Möbius transformation from the unit disk to itself is a conformal map

$$\phi(z) = e^{i\theta}\frac{z - z_0}{1 - \bar{z}_0 z}.$$

Theorem (Riemann Mapping)

Any metric topological disk can be conformally mapped to the unit disk, the mapping is unique up to a Möbius transformation.

Holomorphic 1-forms

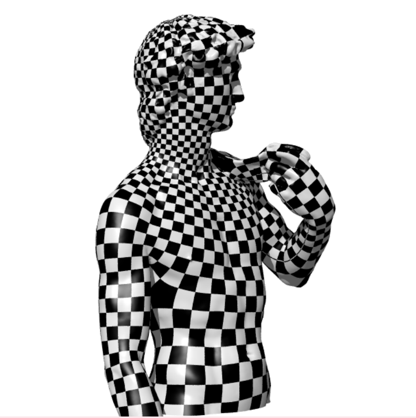
Definition (Holomorphic 1-form)

Suppose Σ is a Riemann surface, $\{z_\alpha\}$ is a local complex parameter, a holomorphic 1-form ω has a local representation as

$$\omega = f(z_\alpha)dz_\alpha,$$

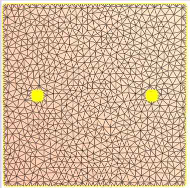
where $f(z_\alpha)$ is a holomorphic function.

Locally, ω is the derivative of a holomorphic function. Globally, it is not.



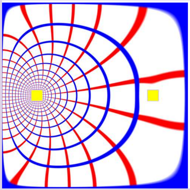
Holomorphic 1-forms

Original Surface



Holomorphic 1-forms

One basis holomorphic 1-form



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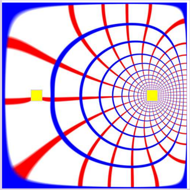
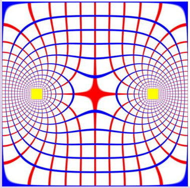


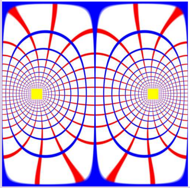
Holomorphic 1-forms

Summation of ω_1 and ω_2



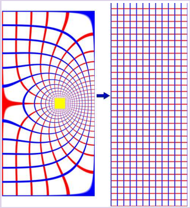
Holomorphic 1-forms

Difference between ω_1 and ω_2



Holomorphic 1-forms

Holomorphic 1-form induces a conformal parameterization.



Holomorphic 1-forms

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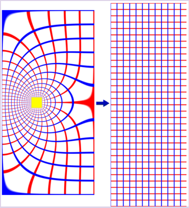
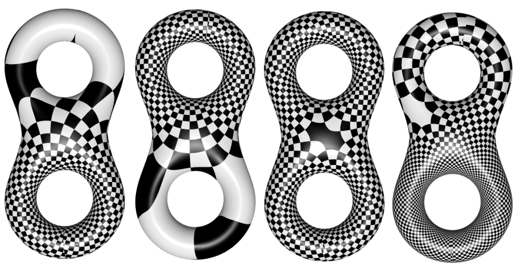


Holomorphic 1-forms

Theorem (Holomorphic 1-forms)

All holomorphic 1-forms form a linear space $\Omega(\Sigma)$ which is isomorphic to the first cohomology group $H^1(\Sigma,\mathbb{R})$.

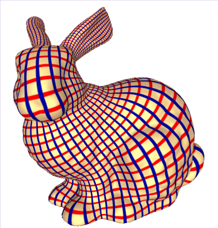


Holomorphic 1-forms

Holomorphic 1-form ω can be treated as two real 1-forms $\omega = (\omega_0, \omega_1)$.

Furthermore, we can treat each 1-form as a vector field, such that

1. $curl\,\omega_0 \equiv 0$
2. $div\,\omega_0 \equiv 0$
3. $\omega_1 = \mathbf{n} \times \omega_0$, where **n** is the normal field.



Holomorphic 1-forms

Intuition Hodge star operator rotates a vector field about the normal a right angle.

Definition (Hodge Star)

Hodge star operator is defined in the following:

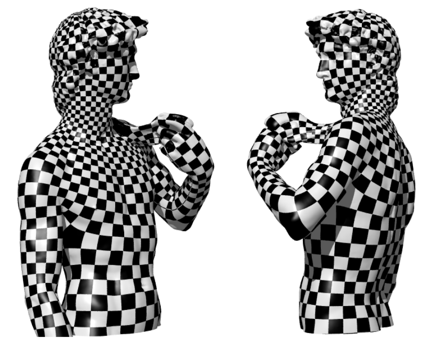
$$*dx = dy, *dy = -dx,$$

Definition (harmonic 1-form)

Suppose Σ is a Riemann surface, ω is differential 1-form, locally ω is the derivative of a harmonic function. Symbolically,

$$d\omega = 0, *d*\omega = 0.$$

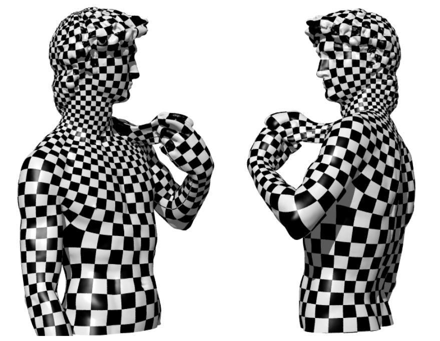
Globally , such harmonic function doesn't exist



Holomorphic 1-forms

Theorem (Hodge)

Each cohomologous class has a unique harmonic 1-form.



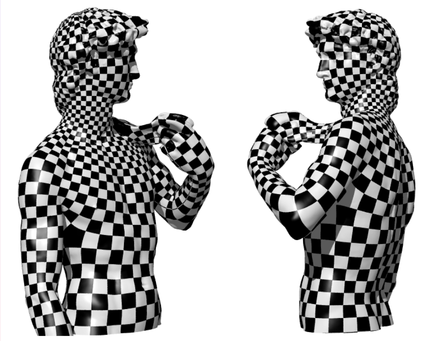
Holomorphic 1-forms

Algorithm for Holomorphic 1-forms

Input : A triangle mesh Σ.
Output : Basis for holomorphic 1-forms

1. Compute cohomology basis $\{\omega_1, \omega_2, \cdots, \omega_n\}$.
2. Heat flow to deform ω_i to harmonic 1-forms.
3. Compute hodge star of ω_i's.
4. return holomorphic 1-form basis

$$\{\omega_1 + \sqrt{-1} * \omega_1, \omega_2 + \sqrt{-1} * \omega_2, \cdots, \omega_n + \sqrt{-1} * \omega_n\}$$



Holomorphic 1-forms

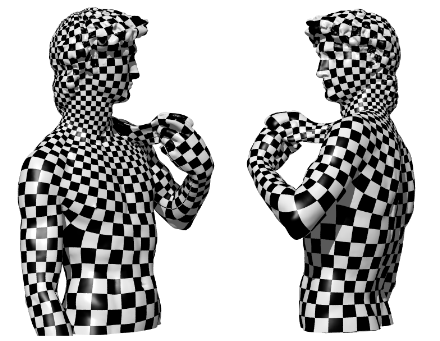
Heat Flow for 1-forms

Suppose $\omega : \{Edges\} \to \mathbb{R}$ is a closed 1-form. Let $f : \{Vertices\} \to \mathbb{R}$ is a function, then

$$f - = \Delta(\omega + df) \times \varepsilon,$$

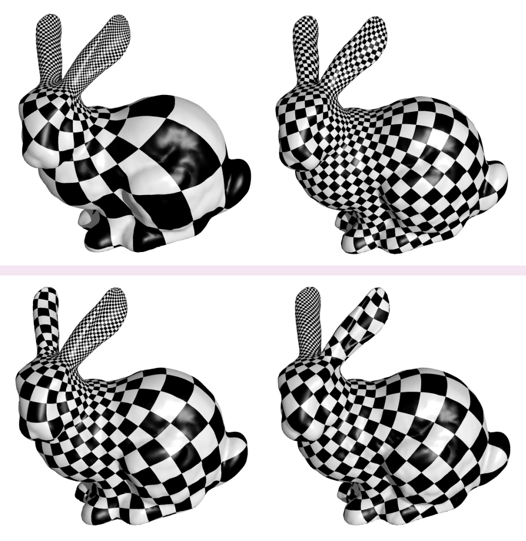
where $\Delta(\omega + df)(v_i)$

$$\sum_{e_{ij}} w_{ij}(\omega(e_{ij}) + f(v_j) - f(v_i)).$$



Holomorphic 1-forms

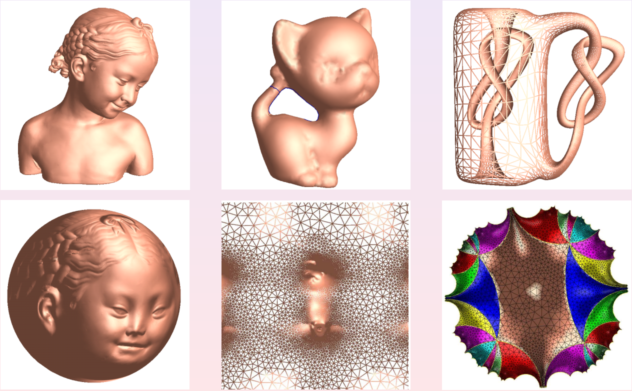
Choose the best cohomology class to optimize the distortion,



Uniformization

Theorem (Poincaré Uniformization Theorem)

Let $(\Sigma, \mathbf{g})$ be a compact 2-dimensional Riemannian manifold. Then there is a metric $\tilde{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ conformal to $\mathbf{g}$ which has constant Gauss curvature.



Spherical Euclidean Hyperbolic

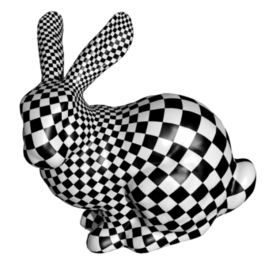
Conformal Metric

Definition

Suppose Σ is a surface with a Riemannian metric,

$$\mathbf{g} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

Suppose $\lambda : \Sigma \to \mathbb{R}$ is a function defined on the surface, then $e^{2\lambda}\mathbf{g}$ is also a Riemannian metric on Σ and called a conformal metric. $e^{2\lambda}$ is called the conformal factor.



Angles are invariant measured by conformal metrics.

Curvature and Metric Relations

Suppose $\bar{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ is a conformal metric on the surface, then the Gaussian curvature on interior points are

$$\bar{K} = e^{-2\lambda}(-\Delta\lambda + K),$$

geodesic curvature on the boundary

$$\bar{k_g} = e^{-\lambda}(\partial_n\lambda + k_g).$$

Definition (Surface Ricci Flow)

A closed surface with a Riemannian metric $\mathbf{g}$, the Ricci flow on it is defined as

$$\frac{dg_{ij}}{dt} = -Kg_{ij}.$$

If the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant every where.

Ricci Flow

Theorem (Hamilton 1982)

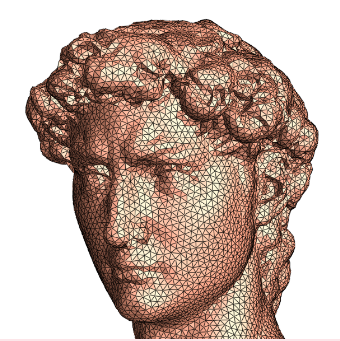
For a closed surface of non-positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) every where.

Theorem (Chow)

For a closed surface of positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) every where.

Generic Surface Model - Triangular Mesh

- Surfaces are represented as polyhedron triangular meshes.
- Isometric gluing of triangles in $\mathbb{E}^2$.
- Isometric gluing of triangles in $\mathbb{H}^2, \mathbb{S}^2$.



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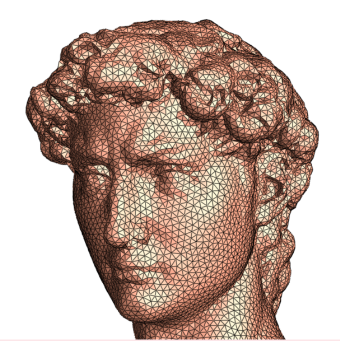

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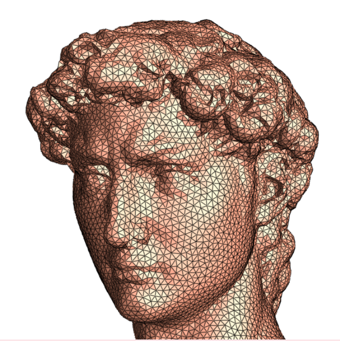
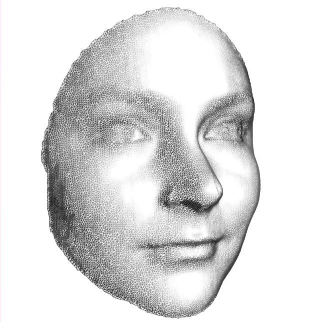


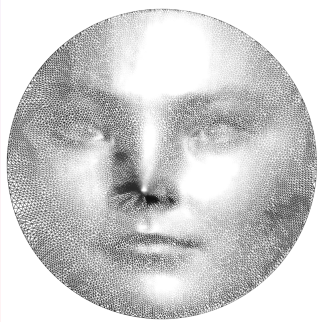
Discrete Metrics

Definition (Discrete Metric)

A Discrete Metric on a triangular mesh is a function defined on the vertices, $l : E = \{all\ edges\} \rightarrow \mathbb{R}^+$, satisfies triangular inequality.

A mesh has infinite metrics.



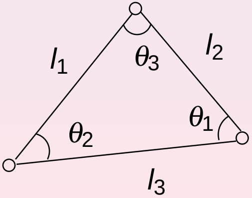


Discrete Metrics

Metric

- Discrete Metric: $l : E = \{all\ edges\} \to \mathbb{R}^1$, satisfies triangular inequality.
- Metrics determine curvatures by cosine law.

$$\cos\theta_i = \frac{l_j^2 + l_k^2 - l_i^2}{2 l_j l_k}, l \neq j \neq k \neq i$$



Derivative Cosine Law

Theorem (Derivative Cosine Law)

Consider an Euclidean triangle $\theta_i = \theta_i(l_1, l_2, l_3)$, $i \neq j \neq k \neq i$, *then*

$$\frac{1}{\sin\theta_i}\frac{\partial\theta_i}{\partial l_j} = \frac{1}{\sin\theta_j}\frac{\partial\theta_j}{\partial l_i}$$

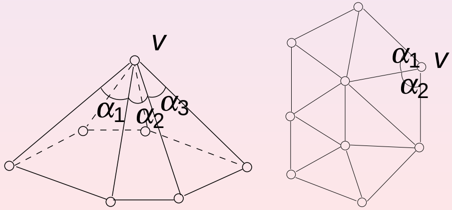
Definition (Discrete Curvature)

Discrete curvature: $K : V = \{vertices\} \rightarrow \mathbb{R}^1$.

$$K(v) = 2\pi - \sum_i \alpha_i, v \notin \partial M; K(v) = \pi - \sum_i \alpha_i, v \in \partial M$$

Theorem (Discrete Gauss-Bonnet theorem)

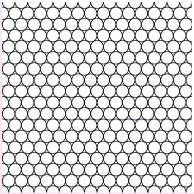
$$\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi\chi(M).$$

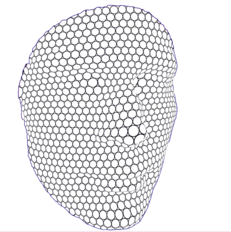


Conformal metric deformation

Conformal maps Properties

- transform infinitesimal circles to infinitesimal circles.
- preserve the intersection angles among circles.





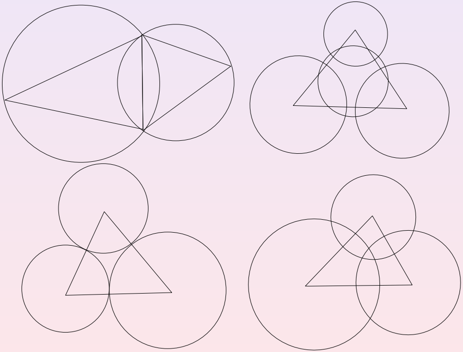
Idea - Approximate conformal metric deformation

Replace infinitesimal circles by circles with finite radii.

Different Circle Patterns

Circle Patterns

There are many local settings for circle patterns. The radius is variable, the intersection angles do not change.



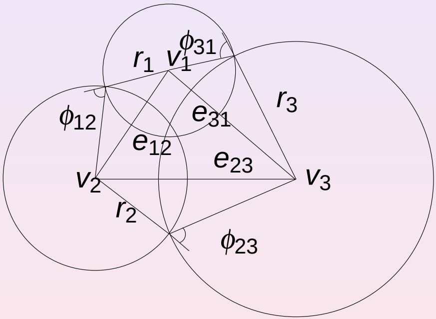
CP Metric

We associate each vertex v_i with a circle with radius γ_i. On edge e_{ij}, the two circles intersect at the angle of Φ_{ij}. The edge lengths are

$$l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j\cos\Phi_{ij}$$

CP Metric (Σ, Γ, Φ), Σ triangulation,

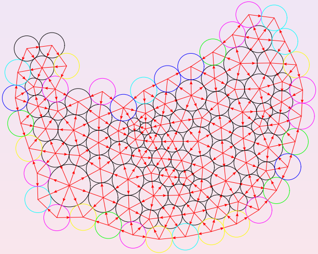
$$\Gamma = \{\gamma_i | \forall v_i\}, \Phi = \{\phi_{ij} | \forall e_{ij}\}$$

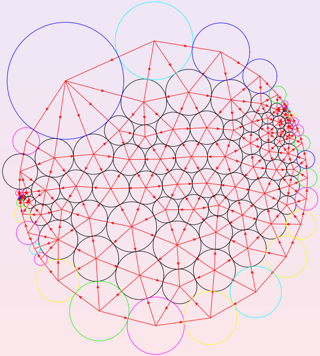


Conformal Equivalent Circle Packing Metrics

Definition (Conformal Equivalent Circle Packing Metrics)

Two circle packing metrics of the same mesh M, $\{M,\Gamma_1,\Phi_1\}$ and $\{M,\Gamma_2,\Phi_2\}$, are *conformal equivalent*, if Φ_1 equals to Φ_2.





Conformal Metric Space

Suppose the vertex set of the mesh is $\{v_1, v_2, \cdots, v_n\}$, we represent a conformal circle packing metric by $\mathbf{u} = (u_1, u_2, \cdots, u_n)$, where $u_i = \log \gamma_i$.

Definition (Normalized Conformal Circle Packing Metric Space)

Each conformal equivalence class of circle packing metrics form a space, we call it *conformal circle packing metric space*. Because scaling doesn't affect curvature, we require $\sum_i u_i = 0$. All such $\mathbf{u}$ form a hyper-plane in the $\mathbb{R}^n$, denoted as Π_u. We call Π_u the *normalized conformal circle packing metric space*.

Curvature Space

Definition (Discrete Curvature Space)

We use $\mathbf{k} = (k_1, k_2, \cdots, k_n)$ to represent the curvature on the vertices of the mesh. Then all such $\mathbf{k}$ form the *discrete curvature space*, which is on a hyper-plane in $\mathbb{R}^n$, $\sum_i k_i = 2\pi\chi(M)$, $\chi(M)$ is the Euler number of the mesh.

Curvature Map

Definition (Discrete Curvature Map)

The discrete curvature Equation defines a discrete curvature map

$$K : \mathbf{u} \to \mathbf{k}. \tag{1}$$

Image of Curvature Map

Given any subset $I \subset V$, let F_I be the set of all faces in M whose vertices are in I and let the link of I, denoted by $Lk(I)$, be the set of pairs (e,v) of an edge e and a vertex v so that (1) the end points of e are not in I and (2) the vertex v is in I and (3) e and v form a triangle.

Theorem (Image of Curvature Space)

All possible curvatures functions **k** *induced by a conformal equivalence class of circle packing metrics* $\{M,\Gamma,\Phi\}$*, where* Γ *varies but* Φ *is fixed, form a* $n-1$ *dimensional convex polytope, such that the total curvature satisfies the Gauss-Bonnet theorem and for any proper subset* $I \subset V$*,*

$$\frac{2\pi|I|\chi(M)}{|V|} > -\sum_{(e,v)\in Lk(I)} (\pi - \Phi(e)) + 2\pi\chi(F_I). \qquad (2)$$

The convex polytope is denoted as Ω_k*.*

Theorem (Inverse Curvature Map)

The curvature map K from normalized conformal circle packing metrics space Π_u to the image of curvature map Ω_k is a C^∞ diffeomorphism, furthermore, it is real analytic.
The derivative map $dK : T\Pi_u(\mathbf{u}) \to T\Omega_k(\mathbf{k})$, satisfies the discrete Poisson equation,

$$d\mathbf{k} = \Delta(\mathbf{u})d\mathbf{u}, \tag{3}$$

where $T\Pi_u(\mathbf{u})$ is the tangent space of Π_u at the point $\mathbf{u}$, $T\Omega_k(\mathbf{k})$ is the tangent space of Ω_k at the point $\mathbf{k}$, $\Delta(\mathbf{u})$ is a positive definite matrix when constrained on $T\Pi_u(\mathbf{u})$.

Discrete Euclidean Ricci flow

Definition (Discrete Ricci flow)

A mesh Σ with a circle packing metric $\{\Sigma, \Gamma, \Phi\}$, where $\Gamma = \{\gamma_i, v_i \in V\}$ are the vertex radii, $\Phi = \{\Phi_{ij}, e_{ij} \in E\}$ are the angles associated with each edge, the discrete Ricci flow on Σ is defined as

$$\frac{d\gamma_i}{dt} = (\bar{K}_i - K_i)\gamma_i,$$

where $\bar{K}_i$ are the target curvatures on vertices. If $\bar{K}_i \equiv 0$, the flow with normalized total area leads to a metric with constant Gaussian curvature.

Idea

Metric deformation is driven by curvature.

Theorem (Chow and Luo 2002)

A discrete Euclidean Ricci flow $\{\Sigma, \Gamma, \Phi\} \rightarrow \{M, \bar{\Gamma}, \Phi\}$ *converges.*

$$|K_i(t) - \bar{K}_i| < c_1 e^{-c_2 t},$$

and

$$|\gamma_i(t) - \bar{\gamma}_i| < c_1 e^{-c_2 t},$$

where c_1, c_2 *are positive numbers.*

Variational Euclidean Ricci flow

Definition

Let $u_i = ln\gamma_i$, the Ricci energy is defined as

$$f(\mathbf{u}) = \int_{\mathbf{u}_0}^{\mathbf{u}} \sum_{i=1}^{n} (K_i - \bar{K}_i) du_i,$$

where $\mathbf{u} = (u_1, u_2, \cdots, u_n)$, $\mathbf{u}_0 = (0, 0, \cdots, 0)$.

Derivative Euclidean Cosine Law

Theorem (Ricci Energy)

Euclidean Ricci energy is Well defined and convex, namely, there exists a unique global minimum.

Proof.

In an Euclidean triangle, with angles $(\theta_1, \theta_2, \theta_3)$ and radius $(\gamma_1, \gamma_2, \gamma_3)$, let $u_i = ln\gamma_i$, according to Euclidean cosine law,

$$\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i}.$$

Therefore $\omega = \sum \theta_i du_i$ is a closed 1-form. The Euclidean Ricci energy is well defined. Direct computation verifies that Hessian matrix is positive definite. □

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Newton's method for Euclidean Ricci energy

Gradient descent Method

Ricci flow is the gradient descent method for minimizing Ricci energy,

$$\nabla f = (K_1 - \bar{K}_1, K_2 - \bar{K}_2, \cdots, K_n - \bar{K}_n).$$

Newton's method

The Hessian matrix of Ricci energy is

$$\frac{\partial^2 f}{\partial u_i \partial u_j} = \frac{\partial K_i}{\partial u_j}.$$

Newton's method can be applied directly.

Algorithm : uniform flat metric for closed surfaces

Ricci Flow for Uniform Flat Metric

Suppose Σ is a closed genus one mesh,

1. Compute the circle packing metric (Γ, Φ).
2. Set the target curvature to be zero for each vertex
$$\bar{K}_i \equiv 0, \forall v_i \in V$$
3. Minimize the Euclidean Ricci energy using Newton's method to get the target radii $\bar{\Gamma}$.
4. Compute the target flat metric.

Algorithm : uniform flat metric for open surfaces

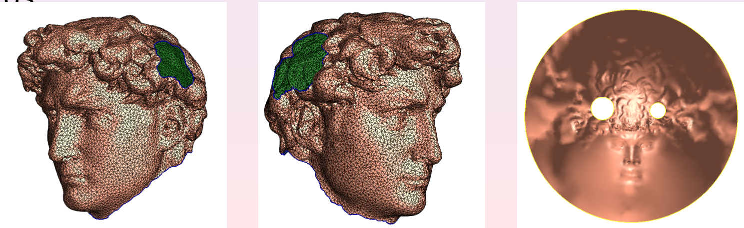
Given a surface Σ with genus g and b boundaries, then it Euler number is

$$\chi(\Sigma) = 2 - 2g - b.$$

Suppose the boundary of Σ is a set of closed curves

$$\partial\Sigma = C_1 \cup C_2 \cup C_3 \cdots C_b.$$

The total curvature for each C_i is denoted as $2m_i\pi, m_i \in \mathbb{Z}$, and $\sum_{i=1}^{b} m_i = \chi(\Sigma)$. The target curvature for interior vertices are zeros.



Algorithm : uniform flat metric for open surfaces

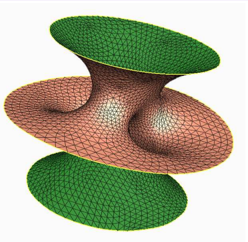
Euclidean Ricci flow for open surfaces

- Use Newton's method to minimize the Ricci energy to update the metric.
- Adjust the boundary vertex curvature to be proportional to the ratio between the current lengths of the adjacent edges and the current total length of the boundary component.
- Repeat until the process converges.

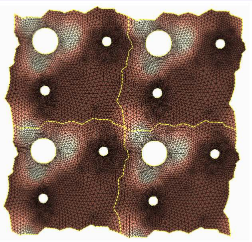
Embedding

1. Determine the planar shape of each triangle using 3 edge lengths.
2. Glue all triangles on the plane along their common edges by rigid motions. Because the metric is flat, the gluing process is coherent and results in a planar embedding.

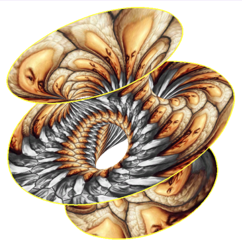
Euclidean Uniform Flat Metric



original surface
genus 1, 3 boundaries

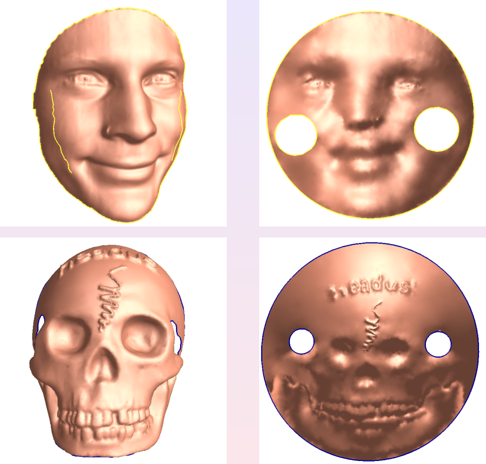


universal cover
embedded in $\mathbb{R}^2$

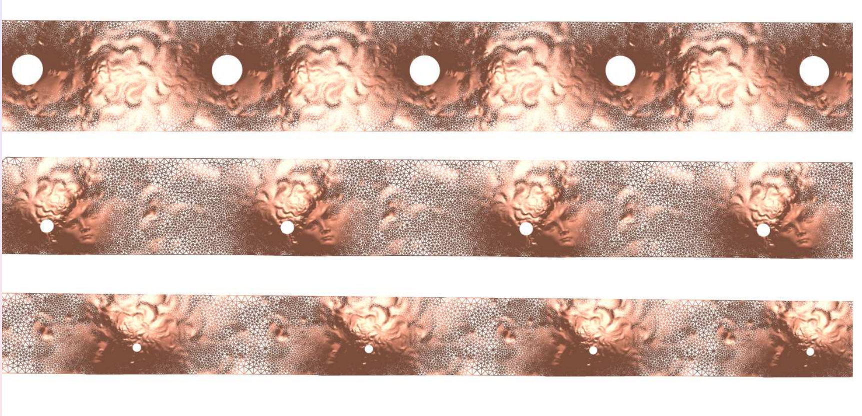


texture mapping

Euclidean Uniform Flat Metric

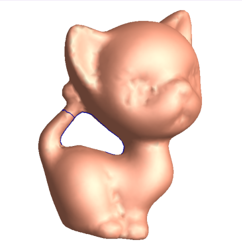


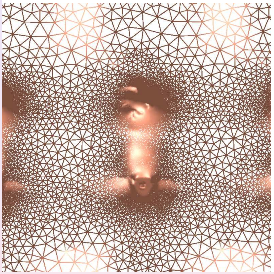
Euclidean Uniform Flat Metric

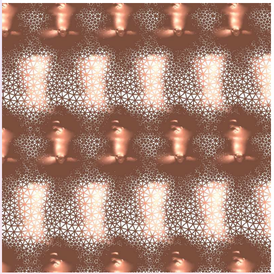


Different boundaries are mapped to straight lines.

Euclidean Uniform Flat Metric


original surface


fundamental domain


universal cover

Optimal Parameterizations Problem

Optimal Conformal Parameterizations

A surface has infinite conformal mappings, different mappings have different area distortions.

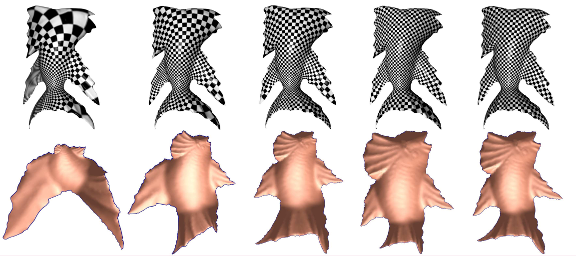
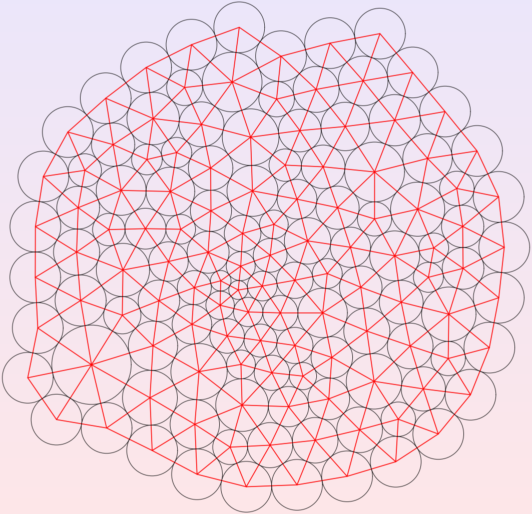
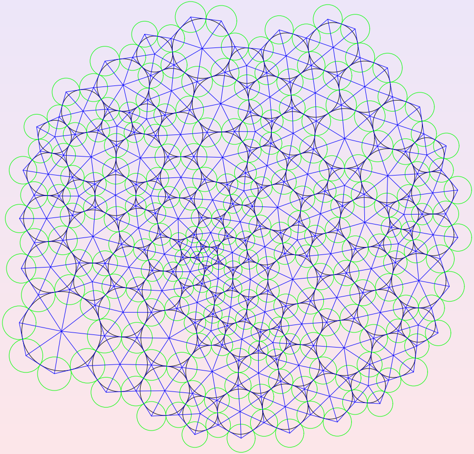


Figure: There are an infinity number of conformal parameterizations of a given surface. We minimize the area distortion within the conformal mappings.

Dual Ricci Flow Method



Dual Ricci Flow Method



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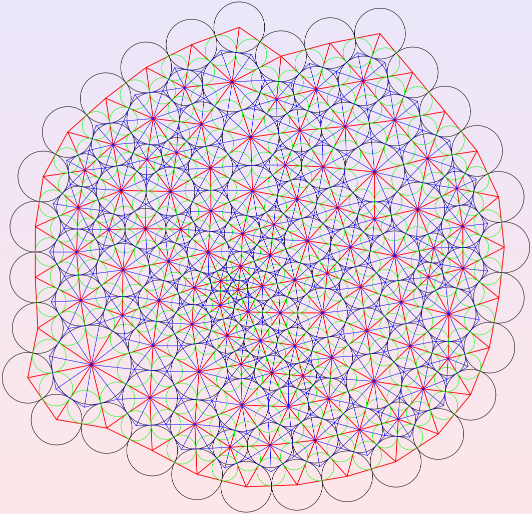
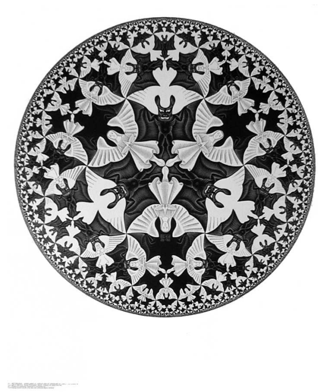


Conformal Model : Poincaré Disk

Poincaré disk

A unit disk $|z| < 1$ with the Riemannian metric

$$ds^2 = \frac{4dzd\bar{z}}{(1-\bar{z}z)^2}.$$

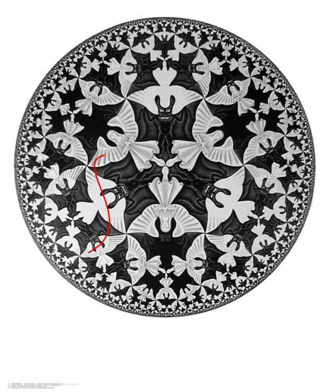


Conformal Model : Poincaré Disk

Poincaré disk

The rigid motion is the Möbius transformation

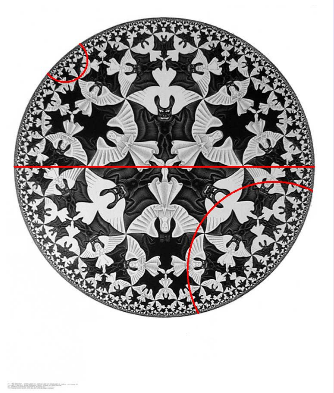
$$e^{i\theta}\frac{z-z_0}{1-\bar{z}_0 z}.$$



Conformal Model : Poincaré Disk

Poincaré disk

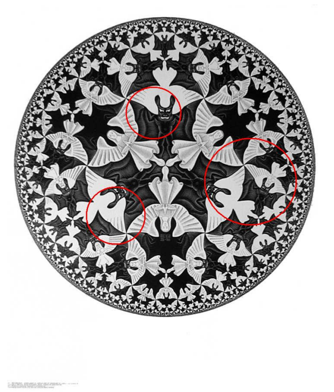
The hyperbolic line through two point z_0, z_1 is the circular arc through z_0, z_1 and perpendicular to the boundary circle $|z| = 1$.



Conformal Model : Poincaré Disk

Poincaré disk

A hyperbolic circle (c,γ) on Poincare disk is also an Euclidean circle (C,R) on the plane, such that $\mathbf{C} = \frac{2-2\mu^2}{1-\mu^2|\mathbf{c}|^2}$, $R^2 = |\mathbf{C}|^2 - \frac{|\mathbf{c}|^2-\mu^2}{1-\mu^2|\mathbf{c}|^2}, \mu = \frac{e^r-1}{e^r+1}$.



Definition (Discrete Hyperbolic Ricci Flow)

Let

$$u_i = ln \tanh \frac{\gamma_i}{2},$$

Discrete hyperbolic Ricci flow for a mesh Σ is

$$\frac{du_i}{dt} = \bar{K}_i - K_i, \bar{K}_i \equiv 0,$$

the Euler number of Σ is negative, $\chi(\Sigma) < 0$.

Discrete Hyperbolic Ricci flow

Theorem (Discrete Hyperbolic Ricci flow, Chow and Luo 2002)

A hyperbolic discrete Ricci flow $(M,\Gamma,\Phi) \to (M,\bar{\Gamma},\Phi)$ *converges,*

$$|K_i(t) - \bar{K}_i| < c_1 e^{-c_2 t},$$

and

$$|\gamma_i(t) - \bar{\gamma}_i| < c_1 e^{-c_2 t},$$

where c_1, c_2 *are positive numbers.*

Discrete Hyperbolic Ricci Energy

Definition (Discrete Hyperbolic Ricci Energy)

The discrete Hyperbolic Ricci energy is defined as

$$f(\mathbf{u}) = \int_{\mathbf{u}_0}^{\mathbf{u}} \sum_{i=1}^{n} (\bar{K}_i - K_i) du_i.$$

Discrete hyperbolic Ricci flow is the gradient descendent method to minimize the discrete hyperbolic ricci energy.

Derivative hyperbolic Cosine Law

Theorem (Hyperbolic Discrete Ricci Energy)

Discrete hyperbolic Ricci energy is well defined and convex, namely, there exists a unique global minimum.

Proof.

In a hyperbolic triangle, with angles $(\theta_1, \theta_2, \theta_3)$ and radius $(\gamma_1, \gamma_2, \gamma_3)$, $u_i = ln \tanh \frac{\gamma_i}{2}$, according to hyperbolic cosine law,

$$\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i}.$$

Therefore $\omega = \sum \theta_i du_i$ is a closed 1-form. The hyperbolic Ricci energy is convex. Direct computation verifies the Hessian matrix is positive definite. □

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Algorithm: Computing Hyperbolic uniformization metric

Hyperbolic Ricci Energy Optimization

1. Set target curvature $K(v_i) \equiv 0$.
2. Optimize the hyperbolic Ricci energy using Newton's method, with the constraint the total area is preserved.

Flattening Mesh in Hyperbolic Space

1. Determine the shape of each triangle.
2. Glue the hyperbolic triangles coherently by Möbius transformation.

Key: all computations use hyperbolic geometry.

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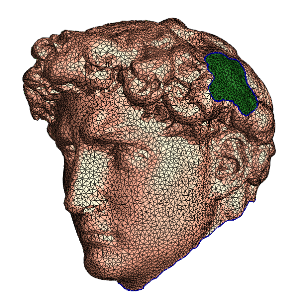
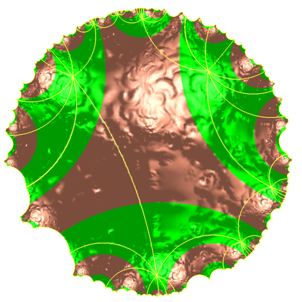
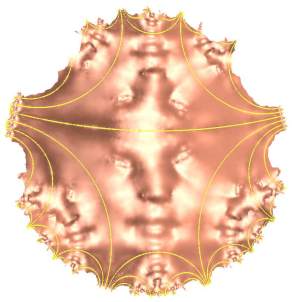




Genus 0 surface with 3 boundaries. The double covered surface is of genus 2. The boundaries are mapped to hyperbolic lines.

Hyperbolic Uniformization Metric



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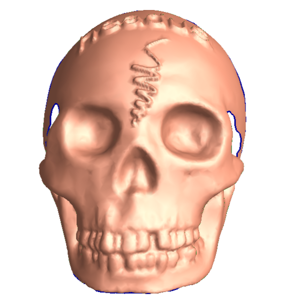

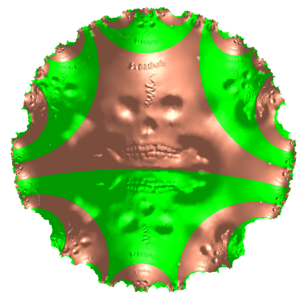

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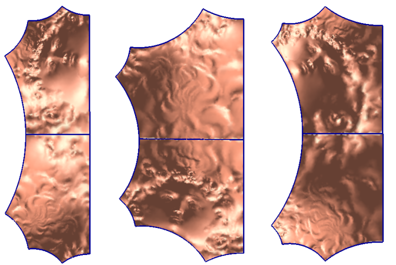
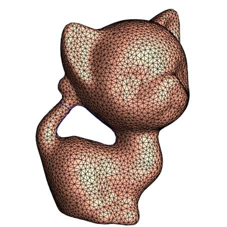
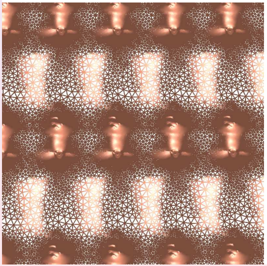
 

Embedding in the upper half plane hyperbolic space model. Different period embedded in the hyperbolic space. The boundaries are mapped to hyperbolic lines.

Universal Covering Space and Deck Transformation





Universal Cover

A pair $(\bar{\Sigma}, \pi)$ is a universal cover of a surface Σ, if

- Surface $\bar{\Sigma}$ is simply connected.
- Projection $\pi : \bar{\Sigma} \to \Sigma$ is a local homeomorphism.

Deck Transformation

A transformation $\phi : \bar{\Sigma} \to \bar{\Sigma}$ is a deck transformation, if

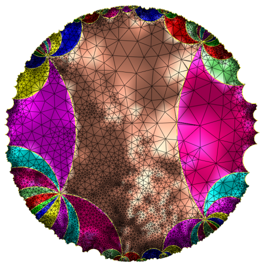
$$\pi = \pi \circ \phi.$$

A deck transformation maps one period to another.

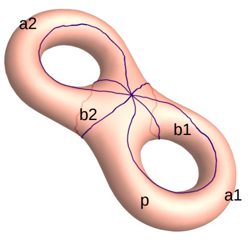
Definition (Funchsian Group)

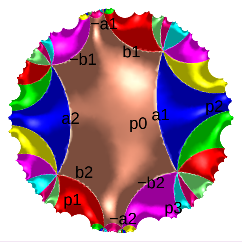
Suppose Σ is a surface, $\mathbf{g}$ is its uniformization metric, $(\bar{\Sigma}, \pi)$ is the universal cover of Σ. $\mathbf{g}$ is also the uniformization metric of $\bar{\Sigma}$. A deck transformation of $(\bar{\Sigma}, \mathbf{g})$ is a Möbius transformation. All deck transformations form the Fuchsian group of Σ.

Fuchsian group indicates the intrinsic symmetry of the surface.



Fuchsian Group





The Fuchsian group is isomorphic to the fundamental group

	$e^{i\theta}$	z_0
a_1	$-0.631374 + i0.775478$	$+0.730593 + i0.574094$
b_1	$+0.035487 - i0.999370$	$+0.185274 - i0.945890$
a_2	$-0.473156 + i0.880978$	$-0.798610 - i0.411091$
b_2	$-0.044416 - i0.999013$	$+0.035502 + i0.964858$

Klein Model

Klein Model

Another Hyperbolic space model is Klein Model, suppose **s**, **t** are two points on the unit disk, the distance is

$$d(s,t) = arccosh \frac{1-\mathbf{s}\cdot\mathbf{t}}{\sqrt{(1-\mathbf{s}\cdot\mathbf{s})(1-\mathbf{t}\cdot\mathbf{t})}}$$

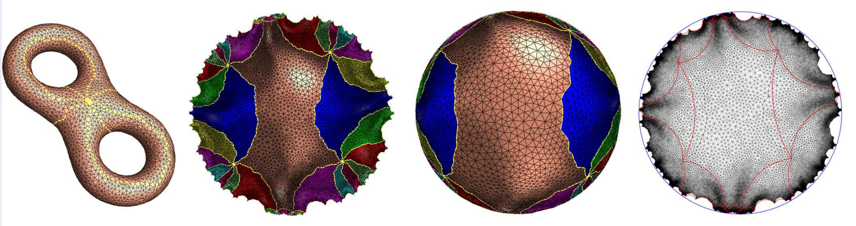
Poincaré vs. Klein Model

From Poincaré model to Klien model is straight froward

$$\beta(z) = \frac{2z}{1+\bar{z}z}, \beta^{-1}(z) = \frac{1-\sqrt{1-\bar{z}z}}{\bar{z}z},$$

Assume ϕ is a Möbius transformation, then transition maps $\beta\circ\phi\circ\beta^{-1}$ are real projective.

Hyperbolic and Real Projective Structure

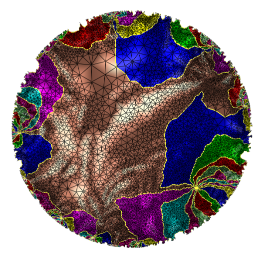


Real projective structure

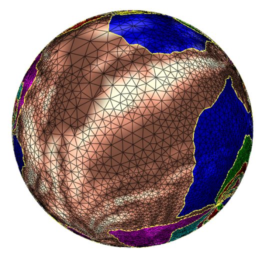
The embedding of the universal cover in the Poincaré disk is converted to the embedding in the Klein model, which induces a real projective atlas of the surface.

Hyperbolic and Real Projective Structure

Surface

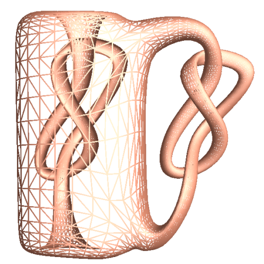


Hyperbolic Structure

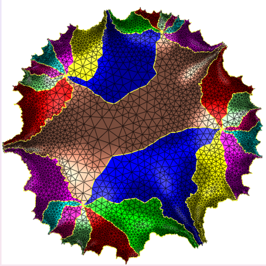


Projective Structure

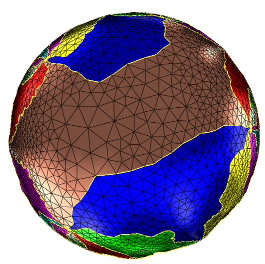
Hyperbolic and Real Projective Structure



Surface, courtesy of Cindy Grimm



Hyperbolic Structure



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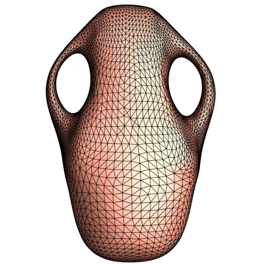
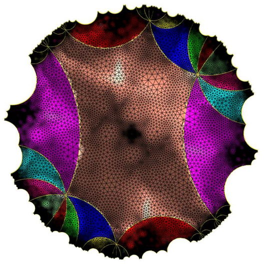


Surface



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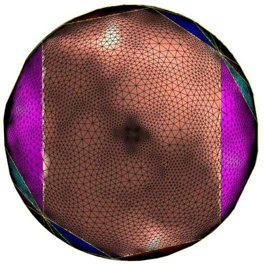

Projective Structure

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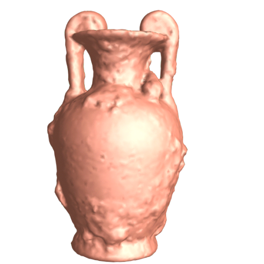

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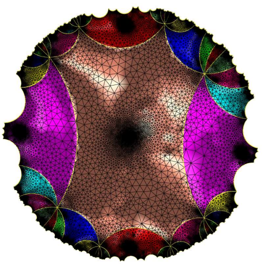

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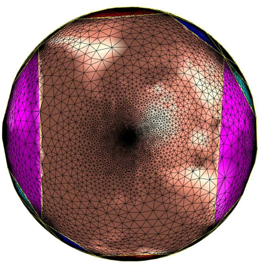
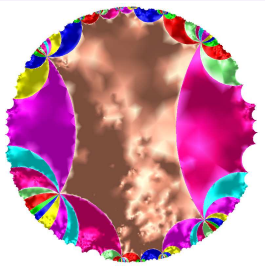
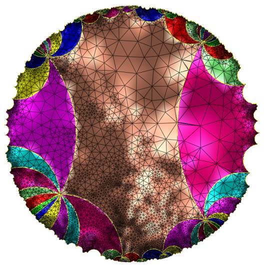
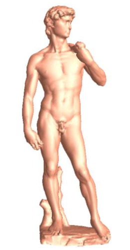
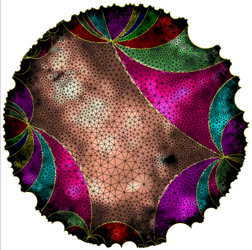
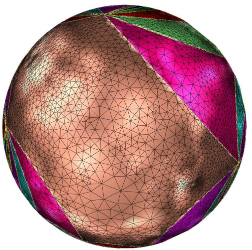


Projective Structure

Hyperbolic Uniformization Metric

Hyperbolic Structure

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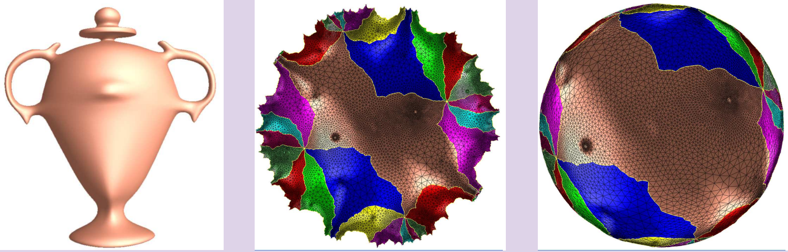
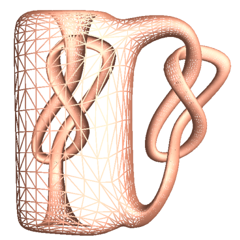
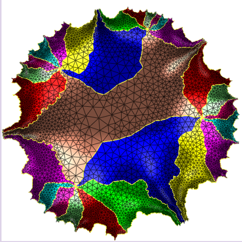


Thanks

For more information, please email to gu@cs.sunysb.edu.

Thank you!