Lecture 9 - Fukaya’s Theorem

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1 Statement

**Theorem 1.1 (Fukaya)** Given \( n, \mu > 0 \), there is a number \( \epsilon \) so that whenever \( N^n, M \) are Riemannian manifolds with \( |\text{sec}| \leq 1, \text{inj}(N) > \mu, \text{and } d_{GH}(N,M) < \epsilon \), then there is a (differentiable?) submersion \( f : M \to N \) so that \((M,N,f)\) is a fiber bundle, the fibers are quotients of nilmanifolds, and \( e^{-\tau(\epsilon)} < |df(\xi)|/|\xi| < e^{\tau(\epsilon)} \).

We use \( \tau \) to indicate a function of \( \epsilon \) with \( \lim_{\epsilon \to 0} \tau(\epsilon) = 0 \). We set up some notation that will be used throughout.

\[
R = \min\{\mu, 1\}/2
\]

\[
\sigma = \text{a small number, } 0 < \epsilon << \sigma << 1
\]

\[
r = \sigma R
\]

2 Embedding into an \( l^2 \) space

Let \((Z,d)\) be a discrete metric space, with \( \epsilon \)-almost isometries into \( M \) and \( N \), \( j_M : Z \to M \) and \( j_N : Z \to N \). Since \( d_{GH}(M,N) < \epsilon \), we can choose \((Z,d)\) and \( j_M, j_N \) so that \( Z = (z_1, \ldots) \) is a countable set, \( M \) (resp. \( N \)) is in the \( \epsilon \)-neighborhood of \( j_N(Z) \) (resp. \( j_M(z) \)), and so that \( j_M(Z) \) (resp. \( J_N(z) \)) is \( \epsilon \)-dense and \( \epsilon/4 \)-separated in \( M \) (resp. \( N \)).

Consider the space \( \mathbb{R}^2 = l^2(Z) \), the Hilbert space on \( Z \). If \( \epsilon \) is small compared to \( \mu \) then we can define \( f_N : N \to \mathbb{R}^2 \) by setting

\[
p \mapsto (\text{dist}_N(p,z_1), \ldots).
\]

This map is 1-1, but not differentiable since \( \text{dist}_N(z_i, \cdot) \) is Lipschitz and not \( C^1 \) (also it is not a map into \( l^2(Z) \) unless \( \#\{Z\} < \infty \)). However we can compose this with a \( C^\infty \)
cutoff function $h : \mathbb{R} \to \mathbb{R}$ that is constant at 0, and equals zero outside a definite radius. Specifically,

\[
\begin{align*}
h(t) &= 1 \quad \text{if } t \leq 0 \\
h(t) &= 0 \quad \text{if } t \geq r \\
h'(t) &\in [-\kappa/r, 0) \quad \text{if } t \in (0, r/8] \cup [7r/8, r) \\
h'(t) &\in [-\kappa/r, -2/r] \quad \text{if } t \in (r/8, 7r/8).
\end{align*}
\]

Now define

\[
f_N(p) = (h(\text{dist}_N(p, z_1)), \ldots).
\]

Let

\[
K = \sup_{x \in N} \# (B_r(x) \cap j_N(Z)).
\]

The following hold, for appropriate constants $C, C_1, C_2$:

- $f_N$ is an embedding
- $\exp^\perp : T^\perp N :\to \mathbb{R}^Z$ is a diffeomorphism out to radius $C\sqrt{K}$.
- (quasi-isometry) we have $|df_N(\xi)|/|\xi| \in (C_1\sqrt{K}, C_2\sqrt{K})$
- If $d_N(x, y)$ is small enough compared to $\epsilon, \sigma, \text{ and } \mu$, then
  \[
d(x, y) \leq CK^{-1/2} \text{dist}_{\mathbb{R}^Z}(f_N(x), f_N(y)).
\]


We would like to say something about a similar map $M \to \mathbb{R}^Z$, but we cannot expect the distance functions $\text{dist}_M(z_i, \cdot)$ can themselves ever be made differentiable. Yet we can smooth them. For $p \in M$ set

\[
d_z(p) = \int_{B_r(z)} \text{dist}_M(p, y) \, dy.
\]

Then $d_z$ is $C^1$ (but not $C^2$), for if $\xi \in T_pM$ then

\[
\xi(d_z)(p) = \int_{B_r(z)} \xi(\text{dist}_M(p, y)) \, dy,
\]

and $\xi(\text{dist}_M(p, y))$ is defined almost everywhere.

**Proposition 2.1** The maps $j_N : N \to \mathbb{R}^Z$, $j_M : M \to \mathbb{R}^Z$ are embeddings, and $j_M(M)$ is in the $6\epsilon\sqrt{K}$-tubular neighborhood of $j_N(N)$.  

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We prove the last statement. Since $M$ and $N$ are $\epsilon$-close in the Gromov-Hausdorff sense, we can find a distance function $d$ on $M \sqcup N$ that restricts to the Riemannian distance on $M$ and $N$ respectively, and so that $M$ is in the $\epsilon$-neighborhood of $N$ and vice-versa, and with $\text{dist}(j_M(z_i), j_N(z_i)) < \epsilon$. Let $p$ be any point of $M$ and let $p' \in N$ be a point with $d(p, p') < \epsilon$. Then

$$d(p, j_M(z_i)) \leq d(p', j_N(z_i)) + d(p, p') + d(j_N(z_i), j_M(z_i))$$

$$d(p', j_N(z_i)) \leq d(p, j_M(z_i)) + d(p, p') + d(j_N(z_i), j_M(z_i))$$

so that

$$|\text{dist}_M(p, j_M(z_i)) - \text{dist}_N(p', j_N(z_i))| \leq 2\epsilon.$$

Since $|h'(t)| \leq 2$ we have

$$|h(\text{dist}_M(p, j_M(z_i))) - h(\text{dist}_N(p', j_N(z_i)))| \leq 4\epsilon.$$

Then

$$|f_M(p) - f_N(p')|^2 = \sum_i (h(\text{dist}_M(p, j_M(z_i))) - h(\text{dist}_N(p', j_N(z_i))))^2 \leq 16K\epsilon^2.$$

Using the averaged quantity $d_{z_i}(p)$ in place of $\text{dist}_N(z_i, p)$ changes the estimates by at most $2\epsilon$, so we get the result. \hfill \Box

Now we have a map $f : M \to N$ given by

$$f = f_N^{-1} \circ \pi \circ \exp^{-1} \circ f_M$$

where $\pi$ indicates the projection from the normal bundle of $f_N(N)$ onto $N$.

### 3 $f : M \to N$ is a fiber bundle

We have to prove that $f_M(M)$ is transverse to the fibers of the normal bundle of $f_N(N)$ in $\mathbb{R}^2$. This follows directly from the following proposition.

**Proposition 3.1** Given any $\nu > 0$, one can choose $\epsilon$, $\sigma$ so that the following holds. If $p \in M$ and $p' = f(p)$, then given any $\xi' \in T_pN$ there exists a $\xi \in T_pM$ such that

$$\frac{|df_M(\xi) - df_N(\xi)|}{|df_N(\xi)|} \leq \nu.$$
Let \( l' : [0, t'] \) be a unit-speed geodesic in \( N \) with \( l'(0) = p' \) and \( \frac{d}{dt}l' = \xi' \). Let \( l : [0, t] \) be a geodesic in \( M \) with \( l(0) = p \) and \( \text{dist}_{RZ}(l(t), l'(t')) < \epsilon \). Now let \( l_i' \) be a geodesic from \( j_N(z_i) \) to \( p' \) and let \( l_i \) be a geodesic from a point \( y \in B_{\epsilon}(j_M(z_i)) \) to \( p \). Let \( \theta_i \) be the angle between \( l \) and \( l_i \), and let \( \theta_i' \) be the angle \( l \) and \( l_i' \).

We prove that
\[
\left| \frac{d}{dt} |_{t=0} h(\text{dist}_N(y, p)) - \frac{d}{dt} |_{t=0} h(\text{dist}_N(j_N(z_i), p')) \right| \leq \nu.
\]

We break the proof into two parts; when \( \text{dist}(j_N(z_i), p) < r/8 - \epsilon \) or \( \text{dist}(j_N(z_i), p) > 7r/8 + \epsilon \), and when \( \text{dist}(j_N(z_i), p) \in [r/8 - \epsilon, 7r/8 + \epsilon] \).

In the first case,
\[
\left| \frac{d}{dt} h(\text{dist}_M(y, p)) \right| = h' \cdot \text{dist} \leq \kappa r/8
\]
and
\[
\left| \frac{d}{dt} h(\text{dist}_N(j_N(z_i), p')) \right| = h' \cdot \text{dist} \leq \kappa r/8
\]
so that
\[
\left| \frac{d}{dt} h(\text{dist}_N(y, p)) - \frac{d}{dt} h(\text{dist}_N(j_N(z_i), p')) \right| \leq 2\kappa r/8 < \kappa \sigma/8
\]

Now we consider the second case. By the first variation formula, we have to prove that \( |\theta_i - \theta_i'| \) is small. By Toponogov’s comparison theorem, we have to prove the following.

\[\textbf{Lemma 3.2} \] Given \( \delta > 0, \mu > 0 \), there is a \( \nu \) with the following properties. Given \( \delta R < t_1, t_2 < R \), assume \( l_1 : [0, t_1] \to M \), \( l_2 : [0, t_2] \to M \) are geodesics with \( l_1(0) = l_2(0) = p \), and \( l_1' : [0, t_1'] \to N \), \( l_2' : [0, t_2'] \to N \) are minimal geodesics with \( l_1'(0) = l_2'(0) = p' \) with \( d(l_1'(t_1'), l_1(t_1)) < \nu \), \( d(l_2'(t_2'), l_2(t_2)) < \nu \). If \( \theta \) and \( \theta' \) are the angles formed by \( l_1(0), l_2(0) \) and \( l_1'(0), l_2'(0) \) respectively, then \( |\theta - \theta'| < \mu \).

\[\textbf{Pf} \]
\[
\square
\[
\square
\]