1 Energy ratio improvement

We have called scale-invariant quantity $\frac{1}{\sqrt{\text{Vol } B_p(r)}} \int_{B_p(r)} |Rm|^\frac{n}{2}$ the “energy ratio.” It is convenient to modify this, and consider the quantity

$$\frac{\text{Vol}^\lambda B(r)}{\text{Vol } B_p(r) \int_{B_p(r)} |Rm|^\frac{2}{2}}.$$

Since we have $\text{Ric} \geq \lambda g$, this quantity is more useful in the use of relative volume comparison. If we restrict ourselves to $r \leq 1$ then these quantities are equivalent.

**Lemma 1.1** Assume $M^n$ is an Einstein manifold, $r \leq 1$, and $\int_{B_p(r)} |Rm|^\frac{n}{2} < \delta$. Then there exist numbers $C < \infty$, $\eta > 0$ so that either

$$\frac{\text{Vol}^\lambda B(r/2)}{\text{Vol } B_p(r/2) \int_{B_p(r/2)} |Rm|^\frac{2}{2}} \geq (1 - \eta) \frac{\text{Vol}^\lambda B(r/2)}{\text{Vol } B_p(r/2) \int_{B_p(r/2)} |Rm|^\frac{2}{2}} \leq (1 - \eta) \frac{\text{Vol}^\lambda B(r)}{\text{Vol } B_p(r) \int_{B_p(r)} |Rm|^\frac{2}{2}},$$

or else the annulus $B_p(5r/8) - B_p(3r/8)$ has

$$\frac{\text{Vol } B_p(r)}{\text{Vol } B_p(r/2)} \geq (1 - \eta) \frac{\text{Vol}^\lambda B(r)}{\text{Vol}^\lambda B(r/2)}.$$

**Pf**

If (1) does not hold, then

$$\frac{\text{Vol}^\lambda B(r/2)}{\text{Vol } B_p(r/2) \int_{B_p(r/2)} |Rm|^\frac{2}{2}} \geq (1 - \eta) \frac{\text{Vol}^\lambda B(r/2)}{\text{Vol } B_p(r) \int_{B_p(r)} |Rm|^\frac{2}{2}} \geq (1 - \eta) \frac{\text{Vol}^\lambda B(r/2)}{\text{Vol } B_p(r/2) \int_{B_p(r/2)} |Rm|^\frac{2}{2}}.$$
and we have \( \frac{\text{Vol}^\lambda B(r/2)}{\text{Vol} B_p(r)} \geq (1 - \eta) \frac{\text{Vol}^\lambda B(r/2)}{\text{Vol} B_p(r)} \). On the other hand

\[
\int_{B_p(r) - B_p(r/2)} |\text{Rm}|^2 \leq \left( \frac{1}{1 - \eta} \frac{\text{Vol}^\lambda B(r/2)}{\text{Vol}^\lambda B(r)} \frac{\text{Vol} B_p(r)}{\text{Vol} B_p(r/2)} - 1 \right) \int_{B_p(r/2)} |\text{Rm}|^2.
\]

Bishop-Gromov volume comparison gives \( \frac{\text{Vol}^\lambda B(r/2)}{\text{Vol}^\lambda B(r)} \frac{\text{Vol} B_p(r)}{\text{Vol} B_p(r/2)} \leq 1 \), so therefore

\[
\int_{B_p(r) - B_p(r/2)} |\text{Rm}|^2 \leq \frac{\eta}{1 - \eta} \int_{B_p(r/2)} |\text{Rm}|^2.
\]

The Key Estimate now gives

\[
\int_{B_p(r) - B_p(r/2)} |\text{Rm}|^2 \leq \frac{\eta}{1 - \eta} Cr^{-4} (\text{Vol} B_p(r) - \text{Vol} B_p(r/2)).
\]

Now let \( q \in B_p(5r/8) - B_p(3r/8) \), so that \( B_q(r/8) \subset B_p(r) - B_p(r/2) \). We have

\[
\int_{B_q(r/8)} |\text{Rm}|^2 \leq \int_{B_p(r) - B_p(r/2)} |\text{Rm}|^2 \leq \frac{\eta}{1 - \eta} Cr^{-4} (\text{Vol} B_p(r) - \text{Vol} B_p(r/2)) \leq \frac{\eta}{1 - \eta} Cr^{-4} \text{Vol} B_q(2r) \leq \frac{\eta}{1 - \eta} Cr^{-4} \text{Vol} B_q(r/8) \frac{\text{Vol}^\lambda B_q(2r)}{\text{Vol}^\lambda B_q(r/8)}
\]

so that

\[
\frac{\text{Vol}^\lambda B(r/8)}{\text{Vol} B_q(r/8)} \int_{B_q(r/8)} |\text{Rm}|^2 \leq \frac{\eta}{1 - \eta} C r^{-4} \text{Vol}^\lambda B_q(2r).
\]

With \( r \leq 1 \) we have that there exists a \( C \) so that

\[
\frac{\text{Vol}^\lambda B(r/8)}{\text{Vol} B_q(r/8)} \int_{B_q(r/8)} |\text{Rm}|^2 \leq \frac{\eta}{1 - \eta} C.
\]

Now if \( \eta \) is chosen small enough that \( C \eta / (1 - \eta) < \epsilon_0 \), then \( \epsilon \)-regularity holds and we get

\[
|\text{Rm}_q| \leq C r^{-2} \sqrt{\eta}.
\]

\[\square\]

**Lemma 1.2** The second alternative in Lemma 1.1 does not hold.
The small curvature and almost-volume annulus together imply the existence of a Cheeger-Colding function $\hat{r}$ that has the following properties

\[ \Delta \hat{r}^2 = 8 \]
\[ |\hat{r} - r| \leq \Phi \]
\[ \frac{1}{|\hat{r}^{-1}(a)|} \int_{\hat{r}^{-1}(a)} |\nabla \hat{r} - \nabla r|^2 \leq \Phi \]
\[ |\nabla \hat{r}| \leq C \]
\[ \left| 1 - \frac{|\hat{r}^{-1}(a)|}{|\partial B_p(a)|} \right| \leq \Phi \]
\[ \frac{1}{|\hat{r}^{-1}(a)|} \int_{\hat{r}^{-1}(a)} |I_{\hat{r}^{-1}(a)} - \frac{1}{\hat{r}^2} g_{\hat{r}^{-1}(a)} \otimes \nabla \hat{r}| \leq \Phi. \]

for some $\Phi = \Phi(\eta)$ where $\lim_{\eta \to 0} \Phi = 0$. We can pass to the universal covering space, where the injectivity radius is bounded and we can take a limit. On this space the injectivity radius is bounded, so it is possible to take a limit as $\eta \to 0$. On the limit space the function $\hat{r}$ has $\nabla^2 r = \frac{1}{\hat{r}^2} g$, so the limit is a warped product with level sets of $\hat{r}$ being space forms. This gives $C^{1,\alpha}$-convergence of $\hat{r}$.

Therefore $\hat{r}^{-1}$ converges in the pointwise sense to a space form. Thus the annulus has (almost) the metric structure of an annulus in a Euclidean cone.

Now consider again the Chern-Gauss-Bonnet theorem

\[ \chi(B_p(3r/4)) = \int |Rm|^2 + \int_{\partial B_p(3r/4)} TP \chi. \]

The boundary term converges to the Euclidean boundary term, which is positive. Since the left-hand side is negative due to the F-structure, we have a contradiction. \hfill \Box

**Theorem 1.3 (\(\epsilon\)-regularity)** If $r \leq 1$ and $\int_{B_p(r)} |Rm|^2 \leq \delta$, then for some $\mu > 0$

\[ \sup_{B_p(\mu r)} |Rm|^2 \leq Cr^{-2}. \]

**Pf**

The Key Estimate gives

\[ \int_{B_p(r/2)} |Rm|^2 \leq Cr^{-4} |Vol B_p(r) - Vol B_p(r/2)| \]
\[ \frac{Vol^3 B(r/2)}{Vol B_p(r/2)} \int_{B_p(r/2)} |Rm|^2 \leq C. \]
for some $C$. Lemmas 1.1 and 1.2 give
\[
\frac{\text{Vol}^k B(r^{2^{k-1}})}{\text{Vol} B_p(r^{2^{k-1}})} \int_{B_p(r/2)} |\text{Rm}|^2 \leq C\eta^k.
\]
Choosing $k > \frac{\log(\epsilon_0/C)}{\log(\eta)}$ gives
\[
\frac{\text{Vol}^k B(r^{2^{k-1}})}{\text{Vol} B_p(r^{2^{k-1}})} \int_{B_p(r/2)} |\text{Rm}|^2 \leq \epsilon_0,
\]
whereupon standard $\epsilon$-regularity goes through. \qed