1 The Hausdorff distance

1.1 Basic Properties

Given a bounded metric space $X$, the set of closed sets of $X$ supports a metric, the Hausdorff metric. Whether $X$ is bounded or not, there is a compact, locally compact topology on the space of closed sets. If $A, B \subset X$ are closed sets, define their Hausdorff distance $d_H(A, B)$ to be the number

$$
\inf \{ r \mid B \text{ is in the } r \text{-neighborhood of } A \text{ and } A \text{ is in the } r \text{-neighborhood of } B \}.
$$

We can say this more precisely as follows. We say $B$ is $r$-close to $A$ (or $B$ is in the $r$-neighborhood of $A$) if

$$
B \subset \bigcup_{x \in A} B(x, r).
$$

Then the Hausdorff distance is the infimum of all $r$ such that $B$ is $r$-close to $A$ and $A$ is $r$-close to $B$. There is still another equivalent definition. Given a point $p \in X$ and a closed set $A \subset X$, define

$$
d(p, A) = \inf_{y \in A} \text{dist}(p, y).
$$

Then the Hausdorff distance is

$$
d_H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\}
$$

That is, $d_H(A, B)$ is the farthest distance any point of $B$ is from the set $A$, or the farthest any point of $A$ is from $B$, whichever is greater.
Theorem 1.1 If $X$ is a bounded metric space, the set of closed sets of $X$ is itself a metric space with the Hausdorff metric.

pf We verify the metric space axioms. First, the symmetry of $d_H$ is clear by definition. Second, $d_H$ satisfies the triangle inequality because if $C$ is in the $r$-neighborhood of $B$ and $B$ is in the $s$-neighborhood of $A$, then $C$ is in the $(r+s)$-neighborhood of $A$. Likewise $A$ is in the $(r+s)$-neighborhood of $C$. Thus $d(A, C) \leq d(A, B) + d(B, C)$. Finally $d_H(A, B) = 0$ implies $A \subset \overline{B} = B$, because if $B$ is in every $r$-neighborhood of $A$ then every point of $A$ is a limit point of $B$. Likewise $B \subset \overline{A} = A$. □

If $X$ is not bounded, the metric space axioms continue to hold, but $d_H(A, B)$ could well be infinity.

1.2 Compactness

Denote the closed subset of $X$ by $C(X)$ (or just $C$ for short). Given a closed set $A$ and a number $r$, let $\mathfrak{B}(A, r)$ be the set of all $D \in C$ with $d_H(B, A) < r$. Since $d_H$ is a metric on $C$, we know that the balls $\mathfrak{B}(A, r)$ are open, and form a neighborhood base.

Obviously the balls with rational radius also form a base, so the induces topology on $C$ is first countable. All metric spaces are Hausdorff, so $(C, d_H)$ is Hausdorff. One can state this directly: since distinct closed sets are separated by a finite distance, say $\epsilon$, so the balls of radius, say, $\epsilon/4$ around each is disjoint.

If $X$ is noncompact, then the topology associated to the Hausdorff distance is neither compact nor even locally compact. To see the local noncompactness, simply pick a sequence $x_i \in X$ that has no convergent subsequence, and define the closed sets $X_i$ to be $X_i = \{x_j\}_{j=1}^i$. Given any neighborhood $\mathfrak{N}$ of $X_\infty = \{x_j\}_{j=1}^\infty$, each $X_i \in \mathfrak{N}$.

If $X$ is noncompact, $(C(X), d_H)$ is not even locally compact. For instance if the base space $X$ is nondiscrete (it has the property that, given any point $x \in X$ and any number $\epsilon > 0$, there is a point $y \in X$ with $d(x, y) < \epsilon$), then it is not locally compact. As an example, we will will show that $\mathbb{R}$ is not locally compact. Let $A = [0, \infty)$ be the half-line, and consider its $r$-neighborhood $B(A, r)$ (wlog assume $r < \frac{1}{2}$). Define the $A_i$ inductively by setting $A_0 = A$ and $A_i = A_{i-1} - (i, i+r/2)$. We have $d_H(A_i, A_j) = r/2$ for any $i \neq j$, so there are no Cauchy subsequences, and therefore no convergent subsequences.

In fact, the metric topology on $(C(\mathbb{R}), d_H)$ is not even locally paracompact. There exist closed sets $A$ such that every neighborhood of $A$ contains an uncountable discrete subset.

In sharp contrast we have the following theorem.

Theorem 1.2 If $X$ is compact, then $(C(X), d_H)$ is compact.

pf
Let \( A_i \) be a sequence of open sets. Each \( A_i \) has a \( \frac{1}{j} \)-net consisting of \( < N_j \in \mathbb{N} \) elements (an \( \epsilon \)-net is a maximal discrete \( \epsilon \)-separated subset; the compactness of \( X \) guarantees the existence of the number \( N_j \)). Let \( A^k_i \subset A_i \) be the union of the \( \frac{1}{j} \)-nets in \( A_i \) for \( 1 \leq j \leq k \); note that the cardinality of \( A^k_i \) is at most \( N_1 + \cdots + N_j \).

Fixing \( k \), some subsequence \( A^k_{i_k} \) converges in the Hausdorff topology, to a some discrete set \( A^k \). Since \( A^k_{i_k} \) is \( \frac{1}{k} \)-close to \( A_{i_k} \), we have that, for large \( i_k \), \( A_{i_k} \) is \( \frac{3}{k} \)-close to \( A^k \). We can require that \( A^k_{i_k} \) is a subsequence of \( A^{k+1}_{i_{k+1}} \), which means \( A^k \subset A^{k+1} \). Since \( A^k \) is \( \epsilon \)-close to \( A^k_{i_k} \) for large \( i_k \), and \( A^k_{i_{k+1}} \) is \( \frac{1}{k} \)-close to \( A^{k+1}_{i_{k+1}} \) which is \( \epsilon \)-close to \( A^{k+1} \), we have that \( A^k \) is \( (\frac{1}{k} + 2\epsilon) \)-close to \( A^{k+1} \), any \( \epsilon > 0 \) so that \( A^k \) is \( \frac{1}{k} \)-close to \( A^{k+1} \).

The diagonal subsequence \( A^k_{i_k} \) converges to some set \( A^\infty \), in which each \( A^k \) is \( \frac{1}{k} \)-dense.

A topology does exist on \( \mathcal{C}(X) \) that is both locally compact and compact, regardless of the compactness of \( X \). Let a base for this topology be set of the form \( N_{K,\epsilon}(A) \), where \( K \subset X \) is compact, \( A \subset X \) is closed, and \( \epsilon > 0 \), where we define

\[
N_{K,\epsilon}(A) = \{ B \in \mathcal{C}(X) \mid d_H(A \cap K, B \cap K) < \epsilon \}.
\]

This topology on \( \mathcal{C}(X) \) is called the pointed Hausdorff topology. If \( X \) is compact, it is the metric topology. If \( X \) is noncompact, this topology is not induced by any metric.

## 2 The Gromov-Hausdorff distance

The Gromov-Hausdorff distance was invented by Gromov for the purpose of making precise the notions of “closeness” and “convergence.” Recall that his “Almost Flat Manifold” theorem states that a compact bounded-curvature manifold that is “close” to being a point has a finite normal cover that is “close” to being a nilmanifold. The idea behind the Gromov-Hausdorff distance is not difficult; here is what Gromov himself has to say:

- “Either you have no inkling of an idea or, once you have understood it, the very idea appears so embarrassingly obvious that you feel reluctant to say it aloud...”

- “I knew [of] it [the Gromov-Hausdorff metric] for a long time, but it just seemed too trivial to write. Sometimes you just have to say it.”

The Gromov-Hausdorff distance significantly extends the idea of the Hausdorff distance (and is not equivalent to it). Given two closed subsets \( A \) and \( B \) of any metric space (not necessarily subsets of the same space), we define

\[
d_{GH}(A, B) = \inf_{f,g} d_H(f_A \rightarrow X(A), g_B \rightarrow X(B))
\]

\(^1\)Taken from Cheeger’s lecture ‘Mikhail Gromov: How Does He Do It?’. 
where the notation $f_{A \to X}$ (resp. $g_{B \to X}$) denotes an isometric embedding of $A$ into some metric space $X$ (resp. isometric embeddings of $B$ into $X$) and the infimum is taken over all possible such embeddings.

In general the topology associated to the Gromov-Hausdorff distance is neither locally compact nor locally paracompact. To redress this we define the pointed Gromov-Hausdorff topology. This is a topology on the set of pointed sets (defined to be pairs $(A,p)$ where $A$ is a closed subset of a metric space and $p \in A$). A local base for this topology are the sets of the form $N_{K,\epsilon}(A)$ (where $A$ is closed, $K \subset A$ is compact and $p \in K$, and $\epsilon > 0$); we define $N_{K,\epsilon}(A)$ to be the set of pointed closed sets $(B,q)$ so that there exists a compact subset $J \subset B$, $q \in J$, and so that there are isometric embeddings $f : A \cap K \to X$ and $g : B \cap J \to X$ into some space $X$ so that $f(p) = g(q)$ and the Hausdorff distance satisfies $d_H(f(A \cap K), g(B \cap J)) < \epsilon$.

This topology is locally compact and compact. If the Gromov-Hausdorff topology is restricted to compact closed sets, the Gromov-Hausdorff topology and the pointed Gromov-Hausdorff topology coincide.

3 The Lipschitz, $C^{k,\alpha}$, and $L^{p,k}$ topologies

The Gromov-Hausdorff topology is not suitable for questions of differentiability or even topology, since Gromov-Hausdorff limits can jump differentiable structures, topologies, and even dimensions. For example a sequence of tori can converge to a round sphere, or to a circle or to a point.

Thus the Gromov-Hausdorff topology is completely inadequate when studying Riemannian structures (curvature, etc), and we have to find something sharper. Let $f : M \to N$ be a map between metric spaces. Define the dilation of $f$ to be

$$dil(f) = \sup_{p,q \in M} \left\{ \frac{\text{dist}_N(f(p), f(q))}{\text{dist}_M(p, q)} \right\}.$$  

We allow $dil(f)$ to take values in $[0, \infty]$. We define the Lipschitz distance between compact homeomorphic metric spaces $M, N$ by

$$\text{Lip}(M, N) = \inf_{f : M \to N} \left| \log(dil(f)) \right| + \left| \log(dil(f^{-1})) \right|.$$  

One easily verifies that this is a metric (up to equivalence of isometric metric spaces). If $M$ is compact then the induced topology is locally compact. If $M$ is noncompact, one can define a “local Lipschitz topology,” meaning convergence occurs iff it occurs when restricted to compact subsets of the original metric spaces $M, N$. The convergence is essentially of Lipschitz type: for instance the graphs of $\frac{1}{n} \sin(n \pi t)$ over the unit interval for $n \in \mathbb{Z}$ converge to the unit interval. If one includes Riemannian metrics of type $C^{0,1}$, then the space of Riemannian metrics on a compact manifold $M$ is locally compact and complete in
the Lipschitz topology; this can be seen by examining the sequence of metrics on a chart in $M$ diffeomorphic to a Euclidean ball and applying the Arzela-Ascoli theorem.

It is possible to further refine the Lipschitz topology in the category of Riemannian manifolds. Given a sequence of Riemannian manifolds $(M_i, g_i)$, one says that they converge to $(M, g)$ in the $C^{k,\alpha}$- or $L^{k,p}$-topology if there are homeomorphisms $f : M \to M_i$ such that the following holds: Given any coordinate chart $U \subset M$ with coordinates $\{x^1, \ldots, x^n\}$, with pullback metrics $g_{i,jk} dx^j \otimes dx^k$, the functions $g_{i,jk}$ converge to $g_{jk}$ in the $C^{k,\alpha}$- or $L^{p,k}$-sense.