1 Moser Iteration

With $S_n$ being the Sobolev (=isoperimetric) constant, recall the lemma from last time:

**Lemma 1.1** There exists a $C = C(n)$ so that if $p \geq \frac{n}{2}$ and

$$\left(S_n^{-n} \int_{\Omega} |Rm|^{\frac{2}{p}}\right)^{\frac{p}{2}} \leq \frac{1}{4n\gamma},$$

then

$$\left(S_n^{-n} \int_{\Omega} \phi^{2\gamma} |Rm|^{p\gamma}\right)^{\frac{1}{p\gamma}} \leq C^\frac{1}{p} p^\frac{n}{p} \sup_{\text{supp } \phi} |\nabla \phi|^{\frac{2}{p}} \left(S_n^{-n} \int_{\text{supp } \phi} |Rm|^{p}\right)^{\frac{1}{p}}$$

We can apply this iteratively to obtain a local $C^\infty$ bound.

**Theorem 1.2** There exists a constant $C = C(n)$ so that if

$$\left(S_n^{-n} \int_{\Omega} |Rm|^{\frac{2}{p}}\right)^{\frac{p}{2}} \leq \frac{1}{4n\gamma},$$

then

$$\sup_{B_q(r/2)} |Rm| \leq C(n) r^{-\frac{n}{p}} \left(S_n^{-n} \int_{B_q(r)} |Rm|^{p}\right)^{\frac{1}{p}}$$

**Pf**

Given $r$, let $r_i = \frac{r}{2} (1 + \frac{1}{2^i})$. Let $\phi_i$ be a function with $\phi_i \equiv 1$ inside $B_q(r_i)$, $\phi_i \equiv 0$ outside $B_q(r_{i-1})$, and $|\nabla \phi_i| \leq 2(r_{i-1} - r_i)^{-1} = r^{-1}2^{i+2}$. Putting

$$\Phi_i = \left(S_n^{-n} \int_{B_q(r_i)} |Rm|^{p\gamma_i}\right)^{\frac{1}{p\gamma_i}},$$
we have from lemma 1.1 that

\[
\Phi_{i+1} \leq C r^{-1} \gamma^{-i} (p \gamma^i) \frac{2}{p-1} \gamma^{-i} (4r^{-1} 2^i) 2p^{-1} \gamma^{-i} \Phi_i
\]

\[
= (C r^{-2} p)^{p-1} \gamma^{-i} (4 \gamma)^{p-1} \gamma^{-i} \Phi_i
\]

Iterating, we get

\[
\Phi_{i+1} \leq (C r^{-2} p)^{\frac{1}{p} \sum_{j=0}^i \gamma^{-j}} (4 \gamma)^{\frac{1}{p} \sum_{j=0}^i j \gamma^{-j}} \Phi_0
\]

\[
\leq (C r^{-2} p)^{\frac{1}{p} \sum_{j=0}^\infty \gamma^{-j}} (4 \gamma)^{\frac{1}{p} \sum_{j=0}^\infty j \gamma^{-j}} \Phi_0
\]

We have

\[
\sum_{j=0}^\infty \gamma^{-j} = \frac{1}{1 - \gamma^{-1}} = \frac{n}{2}
\]

\[
\sum_{j=0}^\infty j \gamma^{-j} = \frac{\gamma}{(\gamma - 1)^2} = \left(\frac{n}{2}\right)^2 \frac{1}{\gamma}
\]

so that

\[
\Phi_{i+1} \leq (C r^{-2} p)^{\frac{1}{p} \frac{1}{n} \gamma^{-1} n^{-2}} \Phi_0
\]

\[
= C(n, p) r^{-\frac{2}{p}} \Phi_0.
\]

We therefore have

\[
\lim_{i \to \infty} \Phi_i = \lim_{i \to \infty} \left( S_n^{-n} \int_{B_q(r)} |Rm| r^{\gamma} \right)^{\frac{1}{p} \frac{2}{\gamma}} = \sup_{B_q(r/2)} |Rm|.
\]

We have proven the standard \(\epsilon\)-regularity lemma:

**Theorem 1.3** There exist constants \(\epsilon_0 = \epsilon_0(n, S_n)\) and \(C = C(n, S_n)\) so that

\[
\int_{B_q(r)} |Rm|^{\frac{2}{\gamma}} \leq \epsilon_0
\]

implies

\[
\sup_{B_q(r/2)} |Rm| \leq C r^{-2} \left( \int_{B_q(r)} |Rm|^{\frac{2}{\gamma}} \right)^{\frac{\gamma}{2}}.
\]
2 Kähler geometry

An almost complex structure on a manifold is a tensor $J : T_p M \to T_p M$ such that $J^2 = -1$ (namely, $J(J(X)) = -X$ for all $X \in T_p M$); clearly this resembles multiplication by $i$ in $\mathbb{C}^n$. A manifold is a complex manifold if it has domains $U_\alpha \subset M$ and maps $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ with transition functions $\phi_{\beta \alpha} = \phi_\beta \phi_\alpha^{-1}$ being holomorphic.

A complex manifold automatically carries an almost complex structure: in a coordinate chart $(x^1, y^1, \ldots, x^n, y^n)$ we just define $J \left( \frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial y^i}$ and $J \left( \frac{\partial}{\partial y^i} \right) = -\frac{\partial}{\partial x^i}$. If the transition functions are holomorphic, then this definition is consistent; the preservation of this definition of $J$ is known as the Cauchy-Riemann condition. But on the other hand, when does the existence of an almost complex structure imply that there are charts with holomorphic transition functions? When this is the case, the almost complex structure is said to be a complex structure, or we say that $J$ is integrable. The Newlander-Nirenberg theorem provides the answer. Let

$$N(X,Y) = [JX,JY] - J[JX,Y] - J[X,JY] - [X,Y]$$

be the Nijenhuis tensor. On a complex manifold it is automatic that $N \equiv 0$.

**Theorem 2.1 (Newlander-Nirenberg)** An almost complex manifold is a complex manifold iff $N(X,Y) = 0$ for all smooth vector fields $X$, $Y$.

A metric $g$ on an almost complex manifold is called Hermitian, $J$-Hermitian, or compatible with the almost complex structure if $g(X,Y) = g(JX,JY)$. In that case we can create the Kähler form $\omega$ by setting

$$\omega(X,Y) = g(JX,Y).$$

It is easy to see that the symmetry of $g$ implies the antisymmetry of $\omega$, making it a 2-form. We say that a manifold $(M, J, g)$ is a Kähler manifold if $J$ is integrable and if $\omega$ is a closed 2-form: $d\omega = 0$.

Note that $\omega$ is a real 2-form, meaning $\omega(X,Y) \in \mathbb{R}$ whenever $X, Y$ are real sections of $T^\mathbb{C}M = TM \otimes \mathbb{C}$. If $\eta$ is another real form in the same DeRham cohomology class, then of course $\eta - \omega = d\phi$ for some 1-form $\phi$. However the so-called $\partial\bar{\partial}$-lemma provides more:

$$\eta - \omega = \sqrt{-1} \partial\bar{\partial}\phi$$

for some function $\phi$. If a fixed Kähler form $\omega$ is given, then $\phi$ is often called the Kähler potential for the Kähler form $\eta$. Note that for a given potential $\phi$ the form $\eta = \omega + \sqrt{-1} \partial\bar{\partial}\phi$ is not necessarily the Kähler form of a Riemannian metric, because the associated metric $g_\eta(X,Y) = -\eta(JX,Y)$ (though symmetric) might not be everywhere positive definite.

In a sense, Kähler geometry is the intersection of Riemannian and symplectic geometry. To be more precise, recall that a metric is Kähler if its holonomy is in $U(n)$, Riemannian if
its holonomy is in $SO(2n)$, and symplectic if its holonomy is in $Sp(n)$. Note that

$$U(n) = SO(n) \cap Sp(n),$$

so that a metric with holonomy in both $SO(2n)$ and $Sp(n)$ is Kähler.

3 Elliptic systems on canonical manifolds

3.1 Extremal Kähler manifolds

On any Kähler manifold, we have

$$\triangle \text{Ric} = Rm \ast \text{Ric} + \nabla^2 R.$$

Therefore if the metric is, for instance, CSC Kähler, then we have an elliptic system

$$\triangle \text{Rm} = Rm \ast \text{Rm} + \nabla^2 \text{Ric} \quad (1)$$

$$\triangle \text{Ric} = Rm \ast \text{Ric}. \quad (2)$$

It is known that many Kähler manifolds do not admit CSC (much less Kähler-Einstein) Kähler metrics. A generalization of the CSC condition was proposed by Calabi, who proposed minimizing the functional

$$C(\omega) = \int R^2 \omega^n$$

over metrics is a fixed class.
If the Kähler metric is extremal in the sense of Calabi, then we do not necessarily have constant scalar curvature, but in fact we have $\triangle X = -Ric(X)$ where $X = R, i$. In this case we have the elliptic system

\begin{align*}
\triangle Rm &= Rm * Rm + \nabla^2 Ric \\
\triangle Ric &= Rm * Ric + \nabla X + \nabla X \\
\triangle X &= Ric * X.
\end{align*}

### 3.2 Other cases

There are two other cases of metrics with elliptic systems. The first is the case of metrics with so-called harmonic curvature, namely $Rm_{ijkl,i} = 0$; this is equivalent to the metric being CSC and $W_{ijkl,i} = 0$.

The other is case of 4-dimensional CSC Bach flat metrics, which includes for instance the CSC half-conformally flat metrics. There are higher dimensional generalizations of the Bach tensor, but I don’t know if making them zero yields an elliptic system.