Lecture 15 - Singularities of F-structures II - Removability of Singularities

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1 Characteristic Forms and Transgressions

Let $G$ be Lie group with algebra $\mathfrak{g}$. Let

$$\mathcal{P} : \mathfrak{g}^\otimes k \rightarrow \mathbb{R}$$

be a symmetric invariant polynomial, which is to say, a map that is

1) Symmetric: $\mathcal{P}(\eta_1, \ldots, \eta_i, \ldots, \eta_j, \ldots, \eta_k) = \mathcal{P}(\eta_1, \ldots, \eta_j, \ldots, \eta_i, \ldots, \eta_k)$

2) Invariant: $\mathcal{P} (\text{Ad}_g \eta_1, \ldots, \text{Ad}_g \eta_2) = \mathcal{P}(\eta_1, \ldots, \eta_k)$, and

3) Polynomial: a sum of elementary multilinear maps on $\mathfrak{g}$ of degree $k$,

where $\eta_1, \ldots, \eta_k \in \mathfrak{g}$ and $g \in G$. The derivative of Ad is ad, so letting $g(t) = \exp(t\theta)$ and putting this into (ii) and taking a derivative gives

$$ii') \sum \mathcal{P}(\eta_1, \ldots, [\theta, \eta_i], \ldots, \eta_k) = 0.$$

Now let $M^n$ be a manifold with structure group $G$ (normally $G$ is $O(n)$, $SO(n)$, or $U(n)$), and let $\Omega_i$ be a $\mathfrak{g}$-valued $l_i$-form for $i \in \{1, \ldots, k\}$. We can define a $\sum l_i$-form

$$\mathcal{P}(\Omega_1, \ldots, \Omega_k)$$

in the obvious way (inserting forms to evaluate the $\Omega_i$ to $\mathfrak{g}$, then taking the polynomial). One easily proves that

$$d\mathcal{P}(\Omega_1, \ldots, \Omega_k) = \sum (-1)^{l_1 + \cdots + l_i - 1} \mathcal{P}(\Omega_1, \ldots, d\Omega_i, \ldots, \Omega_k),$$

which is the usual rule for wedge products. If $\theta$ is a $\mathfrak{g}$-valued 1-form (eg. a connection 1-form), then (ii') is
\[ \sum (-1)^{l_1 + \cdots + l_k} P(\Omega_1, \ldots, [\theta, \Omega_1], \ldots, \Omega_k) = 0. \]

Adding (using the multilinearity), we get
\[ d P(\Omega_i, \ldots, \Omega_k) = \sum (-1)^{l_1 + \cdots + l_i - 1} P(\Omega_1, \ldots, d\Omega_i + [\theta, \Omega_1], \ldots, \Omega_k). \]

If \( \theta \) is indeed a connection 1-form then \( D = d + [\theta, \cdot] \), so we get
\[ d P(\Omega_i, \ldots, \Omega_k) = \sum (-1)^{l_1 + \cdots + l_i - 1} P(\Omega_1, \ldots, D\Omega_i, \ldots, \Omega_k). \]

Therefore \( P(\Omega_1, \ldots, \Omega_k) \) is a closed \( (l_1 + \cdots + l_k) \)-form whenever the \( \Omega_i \) are covariant-constant (ie. \( D\Omega_i = 0 \)). If \( \Omega_i = \Omega = d\theta + \frac{1}{2}[\theta, \theta] \) is the curvature 2-form, then \( D\Omega = 0 \). Therefore, assigned to each connection is a curvature 2-form and so a deRham class in \( H^{2k}(M) \). Given a connection \( \theta \) let \( P_{\theta} \) denote the representative 2\( k \)-form.

The question is whether this class is unique. To answer this, let \( \theta_0, \theta_1 \) be two connection 1-forms, and let \( \theta_t = t\theta_1 + (1 - t)\theta_0 \) be the interpolation between them. Corresponding to the connection \( \theta_t \) is the curvature tensor \( \Omega_t = d\theta_t + \frac{1}{2}[\theta_t, \theta_t] \). Since \( \Omega_t \) is a form of even degree, we compute
\[
\frac{d}{dt} P(\Omega_t, \ldots, \Omega_t) = k P(d\Omega_t/dt, \Omega_t, \ldots, \Omega_t)
= k P(d\theta_t - \theta_0 + [\theta_t, \theta_t - \theta_0], \Omega_t, \ldots, \Omega_t)
= k P(D_t(\theta_1 - \theta_0), \Omega_t, \ldots, \Omega_t)
= kd P(\theta_1 - \theta_0, \Omega_t, \ldots, \Omega_t)
\]

where \( D_t \alpha = d\alpha + [\theta_t, \alpha] \). Therefore
\[ P_{\theta_1} - P_{\theta_0} = k d \int_0^1 P(\theta_t - \theta_0, \Omega_t, \ldots, \Omega_t) dt, \]

and so \( P(\Omega_1, \ldots, \Omega_1) \) and \( P(\Omega_0, \ldots, \Omega_0) \) define the same cohomology class. The \( (2k - 1) \)-form \( k \int_0^1 P(\theta_0 - \theta_1, \Omega_t, \ldots, \Omega_t) dt \) is often called a transgression form, and denoted \( TP = TP(\theta_1, \theta_0) \). We have
\[ \frac{\partial}{\partial \theta_1} - \frac{\partial}{\partial \theta_0} + d TP(\theta_1, \theta_0) = 0 \]

2 Characteristic numbers

If \( n \) is even and \( G = SO(n) \), let \( P(\eta_1, \ldots, \eta_{n/2}) \) be the Pfaffian. If a manifold \( M^n \) has structure group \( SO(n) \) on its frame bundle then this defines a characteristic class, the Euler class. Put \( P_\chi = P(\Omega, \ldots, \Omega) \) for the Levi-Civita curvature 2-form \( \Omega \); this defines the Euler class. (Of course an Euler class can be defined on any even-dimensional \( SO(k) \) principle bundle, but we are only concerned with the frame bundle.) Now let \( X \) be a vector field on \( M \).
with isolated zeros. Replace $X$ with $X/|X|$, so $X$ is defined and $C^\infty$ outside a finite number of singular points. At these singular points the index of $X/|X|$ is defined, and the Euler number of $M$ is the sum of the indices of these singular points. Away from the singularities we have a splitting of the tangent bundle into a parallel and orthogonal distribution. If

$$\theta = \begin{pmatrix} A \\ C \\ B \\ D \end{pmatrix}$$

is the corresponding block decomposition of the the Levi-Civita connection $\theta$, then define a new connection

$$\theta' = \begin{pmatrix} A \\ 0 \\ 0 \\ D \end{pmatrix}$$

Needless to say, $A = 0$, since it is an $\mathfrak{o}(1)$-valued 1-form. Since

$$\Omega' = \begin{pmatrix} 0 \\ 0 \\ dD + \frac{1}{2}[D,D] \end{pmatrix}$$

we have that $\mathcal{P}(\Omega', \ldots, \Omega') = 0$ for any symmetric invariant polynomial $\mathcal{P}$ of degree $\frac{n}{2}$. By the previous section, we have “transgressed” $\mathcal{P}_X$ outside the zeros of $X$:

$$\mathcal{P}_X + dT\mathcal{P}_X = 0.$$

Letting $p_i$ be the zeros of $X$ and putting $B(i, \epsilon) = B_{p_i}(\epsilon)$, we have

$$\int_{M - \bigcup_i B(i, \epsilon)} \mathcal{P}_X = \sum_i \int_{\partial B(i, \epsilon)} T\mathcal{P}_X.$$

A classical theorem of Weyl gives that

$$\lim_{\epsilon \to 0} \int_{\partial B(i, \epsilon)} T\mathcal{P}_X = -C Ind_{p_i}(X/|X|)$$

where $C = C(n)$ is a constant. Therefore

$$\chi(M) = \frac{1}{C} \int_M \mathcal{P}_X.$$

In dimension 4 it turns out that $C = 8\pi^2$, and $\mathcal{P}_X$ is a quadratic functional of the Riemann tensor:

$$\chi(M^4) = \frac{1}{8\pi^2} \int_M \frac{1}{24} R^2 - \frac{1}{2} |\text{Ric}|^2 + |W|^2$$

On the other hand, let $\mathcal{P}_\tau = Tr(\Omega \wedge \Omega)$. It can be proven that $\mathcal{P}_\tau = |W^+|^2 - |W^-|^2$ and

$$\tau = \frac{1}{3} p_1 = \frac{1}{12\pi^2} \int_M |W^+|^2 - |W^-|^2$$

where $\tau$ is the signature of the manifold.
3 Characteristic numbers of manifolds with boundary

Assume the boundary of $M$ is $C^\infty$. If $X$ is perpendicular to the boundary, it is easy to modify the Weyl formula to get

$$\chi(M) = \frac{1}{C} \int_M P_X + \frac{1}{C} \int_{\partial M} TP_X.$$ 

On the other hand another term is introduced to the signature formula

$$\tau(M^4) = \frac{1}{12\pi^2} \int_M |W^+|^2 - |W^-|^2 + \frac{1}{12\pi^2} \int_{\partial M} TP + \eta_{\partial M}.$$ 

The functional $\eta_{\partial M}$ is called the $\eta$-invariant. This invariant is defined for any 3-manifold and depends only on the Riemannian structure of $\partial M$ (not how it is embedded as the boundary of $M$). It is additive over disjoint unions.

Using the Hirzebruch L-polynomial a formula for the signature, in terms of the Riemann tensor, can be obtained for any manifold of dimension $4k$. The corresponding signature formula for $4k$-manifolds with boundary has the boundary corrections coming from both a transgression form and an eta-invariant for the boundary $(4k-1)$-manifolds. See the papers of Atiyah-Patodi-Singer for more information.

4 Signatures of the structures $\mathcal{F}_{1,k}$

Let $DT^n$ indicate the solid torus with boundary $T^{n-1}$. Recall that Rong’s non-polarizable structure $\mathcal{F}_{1,k}$ can be considered to be a disk bundle over a 2-torus, or as a solid torus bundle over a circle. As a solid torus bundle, it is

$$\mathcal{F}_{1,k} = [0, 1] \times DT^3 / \sim$$

where $\sim$ identifies $\{0\} \times DT^3$ with $\{1\} \times DT^3$ with the matrix

$$\begin{pmatrix} 1 & k & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

To see that is is a disk bundle over a torus, consider the projection on each solid torus that takes $DT^3$ to its central circle. Since the central circle is mapped to itself isomorphically, this is well-defined globally.

We claim that the signature of this oriented manifold-with-boundary is precisely $k$. To see this, note that there are two homology classes in $H_2(M, \partial M; \mathbb{Z})$, one of which is carried by the central 2-torus, denoted $T$, and the other is fiber, denoted $D$. We claim the intersection form is

$$\begin{pmatrix} \pm k & 1 \\ 1 & 0 \end{pmatrix}.$$
That $D \cdot D = 0$ and $D \cdot T = 1$ are obvious. To see that $T \cdot T = k$ we shall perturb $T$ to another 2-dimensional submanifold $T'$ and show that $T$ and $T'$ intersect transversely in $k$ places, and that the orientations of the intersections are consistent.

Now let $S$ be the meridian circle on the boundary $T^2 \approx \partial \{1\} \times DT^3$. This is identified to the circle $S' \subset \partial \partial \{0\} \times DT^3 \approx T^2$ that wraps around the meridian once and the longitude $k$ times. Let $S(t)$ be a circle in $\{t\} \times DT^3$ with the following property. If $\pi_t : [0, 1] \times DT^3 \rightarrow DT^3$ is the projection onto the second factor, then the image $\pi_t(S(t))$ is a smooth homotopy from the circle $\pi_0(S(0))$ (that wraps around the boundary $1 - k$ times) and the circle $\pi_1(S(1))$ (that wraps around the boundary $1 - 0$ times), and so that halfway through the homotopy $\pi_t(S(t))$, the circle intersects the boundary circle in precisely $k$ points. Now consider the 2-surface-with-boundary $T'$ that $S(t)$ defines in $[0, 1] \times DT^3$. This surface intersects the central cylinder precisely $k$ times. Also, the boundary circle $S(1)$ is identified to the boundary circle $S(0)$ under $\sim$. After identification, we have therefore have $T \cdot T' = \pm k$. It is also clear that $T'$ is smoothly homotopic to $T$.

5 Embedding of $F_{1,k}$ into a collapsed manifold

Let $M$ be a collapsed manifold with a pure F-structure. All singular irremovable singular orbits are (quotients of) 2-tori, denoted say $T$, with an exponential tubular neighborhood isomorphic to one of the $F_{1,k}$. We can prove that there is some $\rho > 0$ so that the injectivity radius for the exponential map off of $T$ is at least $\rho$.

By the Cheeger-Gromov-Fukaya work on N-structures, there is a critical radius $\epsilon$, so that if this exponential map has injectivity radius $< \epsilon$ then this direction is part of an orbit of a larger N-structure. However, Rong proves that on a definite neighborhood of a singular orbit, the N-structure is in fact just the original F-structure. One way to see this is to note that the singular orbit will remain singular. If there is another collapsed direction, then the N-structure must have a 3-dimensional stalk.

Any singular fiber is therefore 2-dimensional and the isotropy killing fields are therefore 1-dimensional. This implies that a singular fiber is isolated. However this is impossible, since the singular fibers of the F-structure are not isolated.

This implies that the normal injectivity radius from the singular locus of the F-structure is at least $\epsilon$.

6 Volume bounds

Let $Z$ be a connected component of the singular locus. Then $T_\rho(Z)$ (or a double cover) is diffeomorphic to the structure $F_{1,k}$. By hypothesis, $T_\rho(Z)$ has very small volume, $|sec| \leq 1$, and boundary diffeomorphic to a nilmanifold. Note that the second fundamental form of
the boundary is controlled.

It is possible to extend $T_\rho(Z)$ is a complete manifold of small volume and controlled curvature. Near infinity we can give $T_\rho(Z)$ the structure of an almost-flat manifold crossed with a half-open interval. Using the Atiyah-Patodi-Singer formula

$$\tau(T_\rho(Z)) = \frac{1}{12\pi^2} \int_M |W^+|^2 - |W^-|^2 + \frac{1}{12\pi^2} \int_{\partial M} TP_\tau + \eta_{\partial M}.$$  

we get that $\eta_{N^3}$, where $N^3 = \partial T_\rho(Z)$ is very small, where $TP_\tau = 0$ because the second fundamental form vanishes, and where $\int |W^+|^2 - |W^-|^2$ is controlled by the (very small) volume and bounded sectional curvature. Therefore $|\tau(M^4)|$ is very small and therefore zero, contradicting that $\tau(T_\rho(Z)) = k$ (unless $k = 0$ and $T_\rho(Z)$ is the trivial disk bundle over the 2-torus).