1 Pure polarized collapse

Assume the (possibly noncompact) manifold \( X \) admits a pure polarized F-structure. This means \( g \) is a locally constant sheaf whose orbits all have the same dimension. On such a manifold we can split the metric into two parts \( g = g' + h \) where \( h \) vanishes on vectors tangent to the orbits and \( g' \) vanishes on vectors perpendicular to orbits. Set

\[
g_\delta = \delta^2 g' + h.
\]

(1)

**Theorem 1.1 (Pure Polarized Collapse)** As \( \delta \to 0 \) the metric \( g_\delta \) collapses everywhere. Also \( \text{dist}_{g_\delta}(p,q) \) decreases with \( \delta \), and the sectional curvature is bounded on any compact set.

**Pf**

We examine the curvature at the point \( p \) by constructing special coordinates near \( p \). Let \( k \) denote the dimension of the orbits. Let \( N^{n-k} \) be any submanifold through \( p \) transverse to the orbits. Given coordinates \( y^1, \ldots, y^{n-k} \) on \( N \) we can extend these coordinate functions to a neighborhood of \( p \) by projecting along the orbits. Finally a \( k \)-torus acts locally on the orbits themselves; the push-forward of a basis of its Lie algebra is an independent Abelian set of Killing fields parallel to the orbits. The Frobenius theorem says we can integrate these to get the remaining coordinate functions \( x^1, \ldots, x^k \) with coordinate fields \( \frac{d}{dx_i} \) equal to the original killing fields. We can choose the origin on any orbit to be its point of intersection with \( N \).

The coordinates field \( \frac{d}{dy_i} = X_i + V_i \) can be decomposed into a part parallel to the orbits \( X_i \) and a part perpendicular to the orbits \( V_i \). Now make the change of coordinates \( u^i = \delta x^i \). Then

\[
g_\delta = \begin{pmatrix}
\left( \frac{d}{dx^1}, \frac{d}{dx^1} \right)_{g_1} & \delta \left( \frac{d}{dx^1}, \frac{d}{dy^1} \right)_{g_1} \\
\delta \left( \frac{d}{dy^1}, \frac{d}{dx^1} \right)_{g_1} & \delta^2 \left( X_i, X_j \right)_{g_1} + \left( V_i, V_j \right)_{g_1}
\end{pmatrix}.
\]

As \( \delta \to 0 \) the metric converges to a warped product metric.
2 Polarized collapse

If the polarization is not pure, it means that the various $U_\alpha$ in the atlas are such that the corresponding pure substructures $G_\alpha$ possibly have different ranks (though the rank is constant on each $U_\alpha$). We have to modify the metric on each $U_\alpha$ separately, and at the same time push the various $U_\alpha$ away from each other.

**Theorem 2.1** If $\mathcal{G}$ is a polarized $F$-structure on the compact manifold $X$, then $X$ admits a sequence of metrics $g_\delta$ so that

1. The manifold $(X, g_\delta)$ collapses
2. $\text{diam}_{g_\delta}(X) < \text{diam}_{g_1}(X)|\log \delta|$
3. $\text{Vol}_{g_\delta}(X) < \text{Vol}_{g_1}(X)|\delta^k|\log \delta|^n$, some $k \geq 1$
4. Sectional curvature $|K|$ is uniformly bounded.

**Pf**
Let $\{(U_i, G_i)\}_{i=1}^N$ be an atlas. Let $f_\alpha : U_\alpha \to [1, 2]$ be a collection of functions, constant on the orbits of $G$, so that $f_i = 1$ in a neighborhood of $\partial U_i$, and so that $\bigcup_i f_i^{-1}(2) = X$. Put $\rho_i = \delta^{\log_2 f_i}$. We start with the metric $g_0 = \log_2(\delta) g$. On $U_i$ we can write $g_0 = g'_i + h_i$, where $g'_i$ is tangent to the orbits of $G_i$ and $h_i$ is perpendicular. Then define $g_1$ by

$$g_1 = \begin{cases} ho_1^2 g'_1 + h_1 & \text{on } U_1 \\ g_0 & \text{on } X - U_1 \end{cases}$$

Proceed inductively. Once $g_{i-1}$ has been chosen, set $g_{i-1} = g'_i + h_i$ on $U_i$ where $g'_i$ is parallel to the orbits of $G_i$ and $h_i$ is perpendicular, and put

$$g_i = \begin{cases} ho_i^2 g'_i + h_i & \text{on } U_i \\ g_{i-1} & \text{on } X - U_i \end{cases}$$

Now (1), (2), and (3) are obvious, where $k = \min \text{rank } G_i$.

We check that sectional curvature is bounded. Let $p \in X$; let $l = \dim O_p$. Let $U_i$ indicate the atlas charts in which $p$ lies. If we work on a normal atlas, we can arrange that $\{G_j\}$ etc, where the structures have rank $l_1 > \cdots > l_s \geq k$. The metric near $p$ is changed
\text{s times, and we will keep track of the changes in curvature as the metric is changed each
time.}

Let \( N^{n-l} \) be a submanifold transverse to the orbits of \( G_j \), and choose coordinates
\((x^1, \ldots, x^l, y^1, \ldots, y^{n-l})\) as before with \( p = (0, \ldots, 0) \), where the coordinate fields \( \frac{d}{dz^i} \) are
just the action fields of \( G_\alpha \) and the \( y^1, \ldots, y^{n-l} \) are constant on the orbits of \( G_\alpha \).

First we scale the coordinates
\[ x^i = \log \delta \cdot x^i \quad y^i = \log \delta \cdot y^i. \]

In the new coordinates, we still have that \( \frac{d\rho_j}{dy^i} = 0 \), but also that
\[ \frac{d^2 \rho_j}{dy^i dy^k} = \frac{1}{\log \delta \log 2} \frac{d^{2} f_j}{dy^i dy^k} \frac{1}{f_j} \delta^{\log_2 f_j} - \frac{1}{\log \delta \log 2} \frac{1}{f_j^2} \delta^{\log_2 f_j} \frac{d^2 f_j}{dy^i dy^k} \]
\[ + \left( \frac{1}{\log 2} \right)^2 \frac{d^{2} f_j}{dy^i dy^k} \frac{1}{f_j^2} \delta^{\log_2 f_j}. \]

Therefore in these coordinates, the functions \( \rho_j' / \rho_j \) and \( \rho_j'' / \rho_j \) are bounded as \( \delta \to 0 \). Since
by the induction assumption the previous metric \( g_{j-1} \) has bounded curvature, so does the
new metric.

\[ \square \]

3 Nonpolarized Collapse

Let \( G \) be an F-structure on the manifold \( M \). We construct what is called a ‘slice polarization.’

3.1 Pure structure

Let \( \Sigma_i \) be the union of orbits of \( G \) of dimension \( i \). Let \( \Sigma_{\epsilon_i} \) denote the set of points of \( \Sigma_i \) a
distance of \( \epsilon_i \) or greater from \( \partial \Sigma_i \) (this is a “thickening” of \( \Sigma_i \)). If \( N \) is any submanifold let
\( \nu(N) \) denote the normal bundle. Let \( S_{\epsilon_i, r_i} \) denote the set \( \{ v \in \nu(\Sigma_{\epsilon_i}) \text{ s.t. } \|v\| < r_i \} \), and let
\( \Sigma_{\epsilon_i, r_i} \) denote the image of \( S_{\epsilon_i, r_i} \) under the exponential map. If \( r_i \) is chosen small enough,
the exponential map is a diffeomorphism.

**Lemma 3.1** There is an invariant metric \( g \) and numbers \( \epsilon_i, r_i \) so that
\[ (1) \bigcup \Sigma_{\epsilon_i, r_i} = M \]
(2) If \( i < j \), then \( \pi_i = \pi_i \circ \pi_j \) on \( \Sigma_{e_i,r_i} \cap \Sigma_{e_j,r_j} \).

\[ \square \]

Now set \( U_i = \Sigma_{e_i,r_i} \). If \( q \in U_i \), then parallel translation from \( q \) to \( \pi_i(q) \) along a geodesic induces an injection \( G_q \to G_{\pi_i(q)} \).

**Lemma 3.2** There exists an inner product \( \langle \cdot, \cdot \rangle_p \) on \( g_p \), the Lie algebra of stalks \( G_p \), that is invariant under the action of \( G_p \) and under the projections \( \pi_i \) whenever \( \pi_i(q) \) is defined.

\[ \square \]

For \( p \in S_{e_i,r_i} \), let \( K^i_p \) be the (not necessarily closed) subgroup of \( G_p \) whose Lie algebra is the orthogonal complement of the isotropy group of \( p \). Set \( K^i_p = \pi^{-1}_i(K_{\pi_i(p)}) \). It follows from the previous lemmas that the assignment \( p \to K^i_p \) is invariant under the local action of \( G_p \).

We can now describe the collapsing procedure. Let \( f_i, \rho_i \) be as before. Fix \( q \) and let \( U_{i_1}, \ldots, U_{i_j}, i_1 < \cdots < i_j \) be the \( U_i \) with \( q \in U_i \). Let \( Z_{i_1} \subseteq \cdots \subseteq Z_{i_j} \) denote the subspaces of \( T_qM \) tangent to the orbits of \( K^i_{\pi_i(q)}, \ldots, K^j_{\pi_j(q)} \). Let \( W_{i_1} \subseteq \cdots \subseteq W_{i_j} \) denote the subspaces \( W_{i_1} = \pi^{-1}_{i_1}(O_{\pi_{i_1}(q)}), \ldots, W_{i_j} = \pi^{-1}_{i_j}(O_{\pi_{i_j}(q)}) \). Note that also \( Z_{i_j} \subseteq W_{i_j} \).

Now let \( g \) be the invariant metric from Lemma 3.1. Set \( g_0 = \log^2 \delta \cdot g \), and write a decomposition for \( g_0 \)

\[ g_0 = g'_1 + h_1 + k_1, \]

corresponding to \( Z_{i_1}, Z_{i_1}^\perp \cap W_{i_1}, W_{i_1}^\perp \). Put

\[ g_1 = \begin{cases} 
\rho^2 g'_1 + h_1 + \rho^{-2}k_1 & p \in U_1 \\
g_0 & \text{otherwise}
\end{cases} \]

Proceed by induction, letting \( g_{i-1} = g'_i + h_i + k_i \) be the decomposition according to \( Z_{i_i}, Z_{i_i}^\perp \cap W_{i_i}, W_{i_i}^\perp \), and putting

\[ g_i = \begin{cases} 
\rho^2 g'_i + h_i + \rho^{-2}k_i & p \in U_i \\
g_{i-1} & \text{otherwise}
\end{cases} \]

First we claim that curvature is bounded as \( \delta \to 0 \). We establish a coordinate system. Let

\[ m_i = \dim \Sigma_i - i = \dim \Sigma_i - \text{rank}_p \Sigma_i, \]

and let \( s^1, \ldots, s^{m_i} \) be coordinates on \( \Sigma_i \) constant on the orbits. Extend these to \( U_{i_1} \) via \( \pi_{i_1} \). Let \( s^{m_i+1}, \ldots, s^{m_2} \), be coordinates on \( U_{i_2} \), constant on the orbits. Extend these to
$U_i \cap U_{ij}$. Proceed in this way, finally getting coordinates $s^1, \ldots, s^{m_{ij}}$ on $U_i \cap \cdots \cap U_{ij}$. Now compliment these coordinates with additional coordinates $t^1, \ldots, t^{n-i_j-m_{ij}}$ that are constant on the orbits of $K_q^{i_j}$ and so that $s^1, \ldots, s^{i_j}, t^1, \ldots, t^{n-i_j-m_{ij}}$ is a complete system that is transverse to the orbits of $F$. Finally let $x^1, \ldots, x^{i_j}$ be coordinates so that $\frac{d}{dx^1}, \ldots, \frac{d}{dx^{i_j}}$ are fields generated by the action of $K_q^{i_j}$.

Now we compute the curvature. First consider the change of metric $g_0 \mapsto g_1$. Relabel the coordinates

\[
\begin{align*}
  z^1 &= s^1 \\
  \vdots \\
  z^{m_{i_1}} &= s^{m_{i_1}} \\
  y^1 &= s^{m_{i_1} + 1} \\
  \vdots \\
  y^{m_{i_j} - m_{i_1}} &= s^{m_{i_j}} \\
  y^{m_{i_j} - m_{i_1} + 1} &= t^1 \\
  \vdots \\
  y^{n-i_j-m_{i_1}} &= t^{n-i_j-m_{i_j}} \\
  x^1 \\
  \vdots \\
  x^{i_j}.
\end{align*}
\]

The orthogonal decomposition of the tangent space given by $Z_{i_1}, Z^\perp_{i_1} \cap W_{i_1}, W^\perp_{i_1}$ roughly corresponds to the selection of the $x, y, z$ coordinates. Working in the $\Sigma_{i_1}$ stratum, $x^1, \ldots, x^{i_1}$ are coordinates on the rank $i_1$ orbits themselves; this roughly corresponds to $Z_{i_1}$. The subspace $Z^\perp_{i_1} \cap W_{i_1}$ is the subspace directly perpendicular to the stratum; this essentially parametrizes the orbits of $F$ not in $\Sigma_{i_1}$, that is, captures the $y$ coordinates, and also captures the remaining $x^k$. Finally $W^\perp_{i_1}$ parametrizes the orbits of $\Sigma_{i_1}$; in fact the coordinate functions $\frac{d}{dx^1}, \ldots, \frac{d}{dx^{i_1}}$ project to zero in this space, or else the action of some of the other strata $\Sigma_{i_2}, \ldots, \Sigma_{i_j}$ would act on the $\Sigma_1$ stratum, which is impossible, and also the action fields are tangent to the orbits and $W^\perp_{i_1}$ is perpendicular to all orbits. Since the $y$ are coordinates on strata and the strata are perpendicular to $W^\perp_{i_1}$, we get that $\frac{d}{dy}$ is perpendicular to $W^\perp_{i_1}$ as well.

Thus we decompose the vectors

\[
\begin{align*}
  \frac{d}{dx} &= b^1_x v_{x,1} + b^2_x v_{x,2} \\
  \frac{d}{dy} &= b^1_y v_{y,1} + b^2_y v_{y,2} \\
  \frac{d}{dz} &= b^1_z v_{z,1} + b^2_z v_{z,2} + b^3_z v_{z,3}
\end{align*}
\]
according to the decomposition $Z_{i_1}, Z_{i_1}^\perp \cap W_{i_1}, W_{i_1}^\perp$. Multiplying the coordinate functions by $\log \delta$, we have again that $|\rho'_{i_1}/\rho_{i_1}|$ and $|\rho'_{i_1}/\rho_{i_1}|$ are bounded. We get the following matrix for $g$.

$$\begin{pmatrix} (b_1^2)^2 + (b_2^2)^2 & b_1^2 b_1^2 + b_2^2 b_2^2 & b_1^2 b_1^2 + b_2^2 b_2^2 \\ b_1^2 b_1^2 + b_2^2 b_2^2 & (b_1^2)^2 + (b_2^2)^2 & b_1^2 b_1^2 + b_2^2 b_2^2 \\ b_1^2 b_1^2 + b_2^2 b_2^2 & b_1^2 b_1^2 + b_2^2 b_2^2 & (b_1^2)^2 + (b_2^2)^2 + (b_3^2)^2 \end{pmatrix}$$

therefore

$$g_1 = \begin{pmatrix} \rho^2(b_1^2)^2 + (b_2^2)^2 & \rho^2 b_1^2 b_1^2 + b_2^2 b_2^2 & \rho^2 b_1^2 b_1^2 + b_2^2 b_2^2 \\ \rho^2 b_1^2 b_1^2 + b_2^2 b_2^2 & (\rho^2 b_1^2)^2 + (b_2^2)^2 & \rho^2 b_1^2 b_1^2 + b_2^2 b_2^2 \\ \rho^2 b_1^2 b_1^2 + b_2^2 b_2^2 & \rho^2 b_1^2 b_1^2 + b_2^2 b_2^2 & \rho^2(b_1^2)^2 + (b_2^2)^2 + (b_3^2)^2 \end{pmatrix}$$

We make the change of coordinates $x \mapsto \rho_{i_1} x, z \mapsto \rho_{i_1}^{-1} z$. In the new coordinates the matrix reads

$$g_1 = \begin{pmatrix} (b_1^2)^2 + \rho^{-2}(b_2^2)^2 & \rho b_1^2 b_1^2 + \rho^{-1} b_2^2 b_2^2 & \rho^2 b_1^2 b_1^2 + b_2^2 b_2^2 \\ \rho^2 b_1^2 b_1^2 + b_2^2 b_2^2 & (\rho^2 b_1^2)^2 + (b_2^2)^2 & \rho^2 b_1^2 b_1^2 + b_2^2 b_2^2 \\ \rho^2 b_1^2 b_1^2 + b_2^2 b_2^2 & \rho^2 b_1^2 b_1^2 + b_2^2 b_2^2 & \rho^2(b_1^2)^2 + \rho^2(b_2^2)^2 + (b_3^2)^2 \end{pmatrix}.$$ 

We must deal with the $\rho^{-1} b_2^2$ term somehow. As we choose $\delta$ differently, the $b_2^2$ (and the other $b_k^2$ for $K = x, y, z, i = 1, 2, 3$) will be different. Let $b_{k, i}$ denote $b_k^2$ in he metric $g_\delta$.

Since the coordinate fields $d/dx^k, 1 \leq k \leq i_1$, are inside of $Z_{i_1}$ to first order, we get

$$\lim_{\delta \to 0} \frac{b_{k}^2(q)}{\rho} = \lim_{\delta \to 0} \frac{b_{x,i}^2(q) - b_{x,0}^2(q)}{\rho - 0} = \frac{1}{\log \delta} \frac{b_{x,i}^2(q) - b_{x,0}^2(q)}{\log \delta - 0} = 0$$

Letting $\delta \to 0$, the limiting matrix is just

$$g_1 = \begin{pmatrix} (b_1^2)^2 & 0 & 0 \\ 0 & (b_2^2)^2 & 0 \\ 0 & 0 & (b_3^2)^2 \end{pmatrix}.$$
To continue, we now focus attention on \( U_{i_2} \). We readjust choice of coordinates, so now 

\[
\begin{align*}
  z^1 &= s^1 \\
  &\vdots \\
  z^{m_{i_2}} &= s^{m_{i_2}} \\
  y^1 &= s^{m_{i_2}+1} \\
  &\vdots \\
  y^{m_{ij}-m_{i_2}} &= s^{m_{ij}} \\
  y^{m_{ij}-m_{i_2}+1} &= t^1 \\
  &\vdots \\
  y^{n-i_j-m_{i_2}} &= t^{n-i_j-m_{i_2}} \\
  x^1 \\
  &\vdots \\
  x^{i_j}.
\end{align*}
\]

One considers the splitting of the tangent space via \( Z_{i_2}, Z_{i_2}^\perp \cap W_{i_2}, W_{i_2}^\perp \), and repeats the computation of the curvature matrix as above.

To see collapse, the idea is that the orbits are almost totally geodesic as the collapsing proceeds. To be specific, let \( q \in M \) and let \( O_q \) be its orbit. Choose \( r \) so that the exponential map on vectors perpendicular to the orbits is a diffeomorphism on vectors of length \( < r \). There is a number \( c \) so that \( \text{dist}(q, \partial T_{r/2}(O_q)) > c \). However there is a closed loop that is noncontractible in \( T_{r/2}(O_q) \) and has length \( < c'\delta \). For \( \delta < c/c' \) this implies there is a noncontractible geodesic in \( T_{r/2}(O_q) \) of length \( < c'\delta \); hence the injectivity radius converges to 0 at \( q \).

### 3.2 Nonpure collapse

If the structure is not pure, then we work on a regular atlas \( U_1, \ldots, U_A \). Over \( U_\alpha \) we have a pure substructure \( G_\alpha \), and we can carry out the procedure above. If we order the atlases so \( G_{1,p} \subset \cdots \subset G_{A,p} \), then the orbit stratification near \( p \) for higher \( U_\alpha \) refines that for lower \( U_\alpha \). We must also modify the cutoff functions \( \rho^\alpha \) to be equal to 1 in some neighborhood of \( \partial U_\alpha \); this way the charts in the atlas are pushed away from each other as well as the strata inside each chart.