1. **What are the minimum requirements for a classical-type physics to exist?**

1.1. **Our formulation of classical physics to date.** We have an extended phase space $T^*M \times \mathbb{R}$ with a Poincare-Cartan 1-form $\eta = p_i dq^i - H dt$. Notice that, when restricted to phase space $T^*M$, the form

$$p_i dq^i = \eta|_{T^*M}$$

is intrinsic, by which I mean, it does not depend on the existence of any Hamiltonian. We then constructed the **canonical 2-form**

$$\omega = d\eta,$$

which we proved was nondegenerate. We proved that the phase flow was in fact just the vortex flow of this 2-form on extended phase space. Again, notice that the form

$$dp_i \wedge dq^i = \omega|_{T^*M}$$

is intrinsic to $T^*M$, meaning is does not depend on the choice of a Hamiltonian. Also, if the phase flow, restricted to $T^*M$ is denoted by $\frac{d}{dt}$, then

$$i_{\frac{d}{dt}} \omega = -dH.$$

Another highly important feature of mechanics is the Poincare integral invariant. In global form this reads

$$\oint_{\gamma_t} \eta = \text{const},$$

where $\gamma$ is some closed path and $\gamma_t$ is its image at time $t$ under the flow. In infinitesimal form this reads

$$\varphi_t^* \omega = \omega$$

where $\varphi_t$ is the flow itself. This means that the Lie derivative of $\omega$ vanishes

$$L_X \omega = 0$$

where $X = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i}$ is the flow field.

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1.2. An attempt to reduce physics to its minimum. It appears we need a cotangent bundle $T^*M$, a 2-form $\omega = dp_i \wedge dq^i$, and a Hamiltonian in order to define a physics. Note that the Hamiltonian itself is \emph{a posteriori}; the cotangent bundle and the 2-form constitute the substratum on which physics is built.

But can we do with even less? Instead of a cotangent bundle $T^*M$ and a canonical 2-form $\omega = dp_i \wedge dq^i$, maybe all we need is an even-dimensional manifold $N^{2n}$ and a nondegenerate 2-form $\omega$. We can still define the flow vector field $X$ (implicitly) by

$$i_X \omega = -dH.$$ 

The last question to ask is whether the flow defined by $X$ preserves the form $\omega$. We compute

$$L_X \omega = d i_X \omega + i_X d\omega = -ddH + i_X d\omega = i_X d\omega.$$ 

We want this to be 0 regardless of which Hamiltonian we chose. Thus we require $d\omega = 0$. It appears we have found the bare minimum for a physics to exist:

1.3. Statement of the minimum required for a physics to exist. For a reasonable “mechanics” to exist, we need an even-dimensional manifold $N^{2n}$ on which exists a non-degenerate 2-form $\omega$, which is \emph{closed}: $d\omega = 0$. Any arbitrary function can be used as a Hamiltonian.

2. Vector fields and diffeomorphisms

2.1. Diffeomorphisms define vector fields. Let $\varphi_t : M \rightarrow M$ be a family of diffeomorphisms that are parameterized by $t$. For each $t$ it is possible to define a vector field $X_t$, which gives the “direction” of the diffeomorphism at each point. In coordinates, a diffeomorphism can be expressed

$$\varphi_t(x^1, \ldots, x^n) = (\varphi^1_t, \ldots, \varphi^n_t)$$ 

where each $\varphi^i_t$ is a function of the coordinates:

$$\varphi^i_t = \varphi^i_t(x^1, \ldots, x^n).$$ 

Fixing the time at $t = 0$, then given any point $p \in M$ we have

$$X(p) = \left. \frac{d\varphi^i_t}{dt} \right|_{t=0} \frac{\partial}{\partial x^i}.$$ 

If we fix the time at $t = t_0$, the formula is a little more complicated, due to the fact that the diffeomorphism has advanced in position

$$X_t(p) = \varphi^i_t \left( \left. \frac{d\varphi^i_t}{dt} \right|_{t=t_0} \right) \cdot \frac{d}{dx^i}.$$ 

Thus a family of diffeomorphism $\varphi_t$ canonically gives rise, at time $t = 0$, to a vector field:

$$X = \left. \frac{d\varphi^i_t}{dt} \right|_{t=0} \frac{\partial}{\partial x^i}.$$
This is often called “differentiating” the family of diffeomorphisms, for obvious reasons. Can this be done in reverse? Does a vector field always give rise to a smooth family of diffeomorphisms?

2.2. Vector fields define diffeomorphisms. The answer is that if \( X \) is a vector field of differentiability class \( C^{0,1} \), then it does. First we state the theorem:

**Theorem 2.1** (Integration of vector fields). Let \( X \) be a vector field of class \( C^{0,1} \) on a manifold. If \( p \in M \), then there is a path \( \gamma : (-\epsilon, \epsilon) \to M \) so that \( \gamma(0) = p \) and \( \dot{\gamma}(t) = X(\gamma(t)) \).

\[ \text{Pf} \]
Let us use local coordinates \( \{x^1, \ldots, x^n\} \) to examine the situation. Then if \( \gamma \) is any path and \( X \) the given vector field, we have

\[ \gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t)) \quad X(p) = X^i(p) \frac{\partial}{\partial x^i}. \]

Also note that

\[ \dot{\gamma}(t) = \frac{d\gamma^i}{dt}(t) \frac{\partial}{\partial x^i}. \]

Therefore solving \( \dot{\gamma}(t) = X \) is therefore equivalent to solving the (nonlinear) system of ODES

\[ \frac{d\gamma^i}{dt}(t) = X^i(\gamma^1(t), \ldots, \gamma^n(t)). \]

The ODE existence theorem states that this has a unique solution if the \( X^i \) are \( C^{0,1} \)-differentiable.

\[ \square \]

3. The Lie Derivative

3.1. Lie derivatives for objects that either push forward or pull back along diffeomorphisms. Let \( X \) be a vector field, and let \( \phi_t : M \to M \) be its associated family of diffeomorphisms. If \( Y \) is another vector field, we define

\[ L_X Y(p) = \lim_{h \to 0} \frac{Y|_p - \phi_{-t*}(Y|_{\phi_t(p)})}{h}. \]

That is, we push \( Y \) forward along the flow, compare it to the \( Y \) that was already defined there, and take a limit. This is called the Lie derivative of \( Y \) along \( X \).

There is no need to stop here: one can take a Lie derivative of any object either pushes forward or pulls back along a diffeomorphism. For instance, if \( \eta \) is an object that pulls back under diffeomorphism, then for \( p \in M \) we define the Lie derivative of \( \eta \) along \( X \), at the point \( p \), to be

\[ L_X \eta(p) = \lim_{h \to 0} \frac{\phi_{-h*}(\eta|_{\phi_h(p)}) - \eta|_p}{h}. \]
3.2. **Mixed objects.** Note the difference in treatment between objects that push forward and those that pull back. Before we unify our treatment of the Lie derivative, notice that, if $\eta$ pulls back, we have

$$L_X \eta = \lim_{h \to 0} \frac{\varphi_h^* \eta - \eta}{h} = \lim_{h \to 0} \frac{\varphi_h^* \eta - \eta}{h}.$$

But then

$$L_X \eta = \lim_{h \to 0} \frac{\varphi_h^* \varphi_h^* \eta - \varphi_h^* \eta}{h} = \lim_{h \to 0} \frac{\eta - \varphi_h^* \eta}{h}.$$

If we make the definition

$$\tilde{\varphi}_h = \varphi_h^* \quad \text{or} \quad \varphi_h^* - \eta,$$

as appropriate, then we can define

$$L_X A = \lim_{h \to 0} \frac{A - \tilde{\varphi}_h A}{h}.$$

Now it is possible to define a Lie derivative on mixed objects also. If $A \in \otimes^{k,l} M$ is some mixed tensor $A = V_1 \otimes \cdots \otimes V_k \otimes \eta_1 \otimes \cdots \otimes \eta_l$, then

$$L_X A = \lim_{h \to 0} \frac{A - \tilde{\varphi}_h A}{h}$$

where

$$\tilde{\varphi}_h(A) = \tilde{\varphi}_h(V_1) \otimes \cdots \otimes \tilde{\varphi}_h(V_k) \otimes \tilde{\varphi}_h(\eta_1) \otimes \cdots \otimes \tilde{\varphi}_h(\eta_l)$$

$$= \varphi_h^*(V_1) \otimes \cdots \otimes \varphi_h^*(V_k) \otimes \varphi_h^*(\eta_1) \otimes \cdots \otimes \varphi_h^*(\eta_l).$$

3.3. **Properties of the Lie derivative.**

4. **Introduction to Symplectic Geometry**

4.1. **Definition of a symplectic manifold.** A 2-form $\omega$ with $d\omega = 0$ that is nondegenerate at every point of an even-dimensional manifold $N$ is called a symplectic form. A manifold $N$ that admits a symplectic form is called a symplectic manifold. The study of such pairs $(N, \omega)$ is the subject of symplectic geometry.

4.2. **Isomorphism between the tangent and cotangent spaces.** Similarities between Riemannian and symplectic geometry abound. One similarity is that a canonical isomorphisms $T_p M \to T_q M$ exist in both cases. We have already discussed this in the Riemannian case (the $\sharp$ and $\flat$ maps). In the symplectic case, we can interpret $\omega$ as a map $\omega : T_p M \to T_p^* M$ by

$$Y \in T_p M \quad \text{maps to} \quad i_Y \omega \in T_p^* M.$$ 

One can prove that this map is an isomorphism, and therefore has an inverse. Unfortunately there is no widely accepted notation for these maps: the $\sharp$ and $\flat$ symbols are reserved for the Riemannian isomorphisms only.
4.3. **Hamiltonian flows.** Let $H : M \to \mathbb{R}$ be any smooth function. Such a function defines a kind of symplectic gradient vector field, usually given by $X_H$ which is defined (implicitly) by

$$i_{X_H} \omega = -dH.$$ 

The vector field $X_H$ is called the *Hamiltonian vector field* defined by $H$. Since any vector field defines a flow, we now have a *Hamiltonian flow*, $\varphi_t$, associated to $H$.

Let us examine how a function $f : M \to \mathbb{R}$ changes along the Hamiltonian flow.

$$L_{X_H} f = X_H(f) = -\omega^{-1}(dH)(\nabla f)$$