Homework 4  
Due Dec 10

Question 1 Define the function

\[ K(\vec{x}, t) = \frac{1}{(2\pi\hbar)^3} \int_{\mathbb{R}^3} \exp\left( i \frac{\vec{p} \cdot \vec{x} - Et}{\hbar} \right) d\vec{p}, \]

where \( E = E(\vec{p}) \) is equal to the energy of a particle of momentum \( p \) (ie, the Hamiltonian). This function is either called the Schrödinger Kernel or the Schrödinger propagator.

Prove that given any function \( \psi_0 \) the function

\[ \psi(\vec{x}, t) = \int_{\mathbb{R}^3} K(\vec{x} - \vec{y}, t - t_0) \psi_0(\vec{y}) d\vec{y} \]

is a solution of the Schrödinger equation with initial condition \( \psi(\vec{x}, t_0) = \psi_0(\vec{x}) \).

For the case of a free particle (no potentials), we have \( E = \frac{1}{2m} |\vec{p}|^2 \). Prove that

\[ K(\vec{x}, t) = e^{-\sqrt{-1} \frac{\vec{x}^2}{2\hbar t}} \left( \frac{m}{2\pi\hbar t} \right)^\frac{3}{2} \exp\left( \sqrt{-1} \frac{m|\vec{x}|^2}{2\hbar t} \right). \]

Question 2. (Spreading of the wave function.) Consider the convolution above, applied to a particle constrained to move in one dimension. Assume the wave function at time 0 is

\[ \psi(x, 0) = \frac{1}{(\pi \xi_0^2)^\frac{1}{4}} \exp(ip_0 x/\hbar) \exp(-x^2/2\xi_0^2). \]

This is a normalized Gaussian, with \( \Delta \psi = \xi_0 \). If time is allowed to vary, prove that

\[ \Delta \psi = \xi_0 \left( 1 + \frac{\hbar^2 t^2}{m^2 \xi_0^4} \right)^\frac{1}{2}. \]

Question 3. (Low dimensional Clifford algebras) Consider the real Clifford algebras \( Cl(n) \). Prove that \( Cl(1) \) is isomorphic to \( \mathbb{C} \), \( Cl(2) \) is isomorphic to \( \mathbb{H} \), and \( Cl(3) \) is isomorphic to \( \mathbb{H} \oplus \mathbb{H} \).

Question 4. (The Dirac Equation) Write down the Dirac equation for a particle in an externally applied electromagnetic field \( \vec{B} = \partial / \partial z \). Solve this equation.