1 Required Knowledge

What are coordinates?

Given a space $M$ and a point $p \in M$, what is $T_p M$, $T^*_p M$?

Given coordinates $\{x^1, \ldots, x^n\}$, what, precisely, does $\frac{\partial}{\partial x^i} \bigg|_p$ mean?

Given a function $f$, what does $df$ mean?

How are vector fields and covector fields expressed?

If $X$ is a vector field and $\omega$ is a covector field, what is $\omega(X)$?

Given a metric $g$ on $V$, how is the metric on $V^*$ defined?

Given a metric $g$ on $V$ with components $g_{ij}$, what are the components $g^{ij}$ of the corresponding metric $g$ on $V^*$?

Let $M$ be a space with metric $g$. If $X$ is a vector field, what is $X_p$? If $\omega$ is a covector field, what is $\omega^i$?

Let $M$ be a space with metric $g$. If $\gamma(\tau)$, $a < \tau < b$ is a path, what is its length?

If $\omega$ is a $p$-form, how is $d\omega$ computed?

What are the four defining properties of the $d$ operator?
2 Practice Problems

1) Consider the Minkowski metric on $\mathbb{R}^{1,1}$. Let $X = x^0 \frac{\partial}{\partial x^0} + x^1 \frac{\partial}{\partial x^1}$ be a vector field, and let $\omega = x^1 dx^0 + x^0 dx^1$ be a 1-form. Find $X^\flat$ and $\omega^\sharp$. Compute $\omega(X)$ and $X^\flat(\omega^\sharp)$.

2) Let $g = 4 \left( (x^1)^2 + 1 \right) dx^1 \otimes dx^1 - x^1 x^2 dx^1 \otimes dx^2 - x^1 x^2 dx^2 \otimes dx^1 + (x^1 x^2)^2 dx^2 \otimes dx^2$ be a 2-tensor on $\mathbb{R}^2$. Is this a metric?

3) Let $g = e^{-(x^1)^2-(x^2)^2} dx^1 \otimes dx^1 + e^{-(x^1)^2-(x^2)^2} dx^2 \otimes dx^2$ be a 2-tensor on $\mathbb{R}^2$. Prove $g$ is a metric. Compute $g^{ij}$. Let $X = \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2}$ and compute $X^\flat$.

4) Let $\{x, y\}$ be coordinates on $\mathbb{R}^2$. Let $\omega = e^{-x^2-y^2} dx$. Compute $d\omega$, then convert $d\omega$ into polar coordinates. Next, convert $\omega$ into polar coordinates first, and then compute $d\omega$. Do you get the same answer?

5) Let $g = \frac{r^2}{(1+r^2)^2} \left( dr \otimes dr + r^2 d\theta \otimes d\theta \right)$ be a 2-tensor. Prove that this is a metric on $\mathbb{R}^2 - \{o\}$. Determine the distance from the origin to the “point at infinity.”