Problem set 2
Due Feb 11

Problem 1) Let \( \{x^0, x^1\} \) be coordinates for \( \mathbb{R}^{1,1} \); this will be the lab frame. A second observer is traveling at \( \frac{3}{5} c \) with respect to the lab, and places \( \{y^0, y^1\} \) coordinates on space-time (with the same origin as the \( \{x^0, x^1\} \) coordinates). A third observer is traveling at \( \frac{12}{13} c \) with respect to the lab, and places \( \{z^0, z^1\} \) coordinates on space-time (with the same origin as the \( \{x^0, x^1\} \) coordinates).

a) Make a sketch of space-time according using the \( \{x^0, x^1\} \) coordinate system. As accurately and neatly as possible, superimpose the \( \{y^1, y^2\} \) axes on your sketch. To do this, first find and label the points

\[ p_0 = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \{y^i\}, \quad p_1 = \left( \begin{array}{c} 0 \\ c \end{array} \right) \{y^i\}. \]

b) On the same graph, superimpose the \( \{z^1, z^2\} \) axes. To do this, first find and label the points

\[ q_0 = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \{z^i\}, \quad q_1 = \left( \begin{array}{c} 0 \\ c \end{array} \right) \{z^i\}. \]

c) Superimpose the light cone and the pseudospheres of radius 1, 2, and 3.
**YOUR GRAPH MUST BE VERY NEAT AND VERY LARGE.

Problem 2) (The Twin Paradox) A traveler leaves the Earth traveling at speed \( v \). After reaching a distance of \( d \) as measured by a stationary observer on Earth, the traveler immediately turns around and returns to Earth at the same speed. According to an Earth-bound observer, how long was the traveler gone? According to the traveler, how much time did the trip take?

Problem 3) (Tachyons) Tachyons are hypothetical particles that travel faster than light. Consider two spaceships leaving the space-time point \( o \), traveling at velocity \( v \) relative to each other. Each spaceship is equipped with a tachyon emitter, which emits a tachyon of velocity \( w \) as measured in the emitter’s rest-frame. At time \( t \), the first ship emits a tachyon, which is received and immediately re-emitted by the second ship, and then received again by the first ship. Let the event \( p \) be the emission of the tachyon from the first ship, let the event \( q \) be the reception and re-emission of the tachyon from the second ship, and let the event \( p' \) be its reception by the first ship.

a) Let \( \{x^0, x^1\} \) be the space-time coordinates of the first ship. Express \( p \) and \( q \) in the \( x^0-x^1 \) coordinate system.

b) Let \( \{\xi^0, \xi^1\} \) be the space-time coordinates of the second ship. Express \( p, q, \) and \( p' \) in the \( \xi^0-\xi^1 \) coordinate system.

c) Express \( p' \) in the \( x^0-x^1 \) coordinate system.

d) Show that if \( w > \frac{v}{\gamma_b} \), the first ship received the tachyon before it was emitted.
**Problem 4** *(The velocity addition formula)* Assume an observer $o'$ travels at speed $v$ with respect to the lab. Of course $o'$ sees the lab moving with speed $-v$. Assume, in addition, that $o'$ sees a particle traveling in the direction opposite the lab’s direction, with speed $w$. Prove that the lab observes the particle moving with speed \[ \frac{v + w}{1 + \frac{vw}{c^2}}. \]

(Hint: You may work in $\mathbb{R}^{1,1}$, instead of the full $\mathbb{R}^{1,3}$. First draw the situation from the perspective of $o'$, then make a transformation from $o'$’s frame back to the lab frame.)

**Problem 5** \((\mathcal{O}(k, n) \text{ is a group})\) In this problem we will prove that the set $\mathcal{O}(k, n)$ is indeed a group. You may consider $\mathcal{O}(k, n)$ to be the matrix group consisting of $(k+n) \times (k+n)$ matrices $A$ so that $A^T I_{k,n} A = I_{k,n}$. Recall that $I_m$ is the $m \times m$ identity matrix, and that the matrix $A^{-1}$ is the inverse of the matrix $A$, if it exists. Prove the following:

a) If $A, B \in \mathcal{O}(k, n)$, then $AB \in \mathcal{O}(k, n)$.

b) The identity matrix $I_{k+n}$ is in $\mathcal{O}(k, n)$.

c) If $A \in \mathcal{O}(k, n)$, then $A^{-1}$ exists and $A^{-1} \in \mathcal{O}(k, n)$.

d) If $A, B, C \in \mathcal{O}(k, n)$, then $(AB)C = A(BC)$.

**Problem 6** Let $V$ be the vector space of the 1-variable functions spanned by $f_1 = \sin(x)$, $f_2 = \cos(x)$, $f_3 = e^x$, and $f_4 = xe^x$. Let $A : V \to V$ be the linear operator $A(\alpha) = \int \alpha \, dx$, where the antiderivative is taken to have zero constant term.

a) Which of the following belong to $V$? Circle all that apply.

- $\cos(x) - e^x + 2xe^x$
- $\sin^2(x)$
- $e^{2x}$

b) Express $A$ as a matrix in the basis $\{f_1, f_2, f_3, f_4\}$.

c) Express $A$ as a matrix in the basis $\{e_1, e_2, e_3, e_4\}$ where $e_1 = \sin(x) + \cos(x)$, $e_2 = \sin(x) - \cos(x)$, $e_3 = e^x$, and $e_4 = 2e^x - xe^x$.

d) Express the function $2\sin(x) + \cos(x) - e^x + xe^x$ in both the $\{f_i\}$ and $\{e_i\}$ bases.