Lecture 25 - The stress-energy-momentum tensor

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1 The Cauchy stress tensor

The Cauchy stress tensor $\vec{T}$ is a construct of classical physics. It is a $(0, 2)$-tensor, and can be written essentially as a $3 \times 3$ matrix:

$$\vec{T} = T_{ab} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}.$$ 

It’s meaning is as follows: if $\hat{n}$ and $\hat{v}$ are unit vectors, then $\vec{T}(\hat{n}, \hat{v})$ is the force communicated across the $\hat{n}$-plane in the $\hat{v}$-direction.

We can interpret the components $T_{ab}$ as follows:

The diagonal elements are the pressures: $T_{11}$, for instance, is the force communicated across the $x^1$-plane in the $x^1$-direction, that is, the pressure across the $x^1$-plane.

The off-diagonal elements are the shear forces: $T_{12}$, for instance, is the $x^2$-force communicated across the $x^1$-plane.

The Cauchy stress tensor has many applications in classical physics and particularly in engineering applications. The notation $\vec{T}$ is meant to indicate that $\vec{T}$ is a classical (tensor) quantity, not meant to indicate that it is a vector.

2 The stress-energy-momentum tensor

The entries in the Cauchy stress tensor depend on the choice of reference frame, and is therefore inadequate for relativistic applications. We require a fully Lorentz-invariant version of this tensor.
First note that force is the same as the time-derivative of momentum:

\[ f = ma = \frac{mdv}{dt} = \frac{dp}{dt} \text{ where } p = mv \text{ is classical momentum} \]

Thus force across a boundary can be regarded as momentum flux across a boundary.

The 4-dimensional stress-energy-momentum tensor (or energy-momentum tensor or stress-energy tensor) is

\[ T = T_{ij} = \begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{pmatrix} \]

The space-components are just the components of the Cauchy stress tensor as seen in the observers rest-frame. The other components have the following interpretation:

- \( T_{00} \) is energy (aka mass) density
- \( T_{0i} \) (\( i \neq 0 \)) is momentum density
- \( T_{0i} \) (\( i \neq 0 \)) is energy (aka mass) flux
- \( T_{ij} = T_{ij} \) (\( i, j \neq 0 \)) are the various momenta fluxes (aka pressures and shear forces).

The stress-energy tensor can be computed from detailed knowledge of the distribution of matter, energy, and forces in some region of space-times.