Lecture 23 - The parallel transport equation, the Riemann curvature tensor, and the Jacobi equation

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1 Parallel transport

Given a path $\gamma(\tau)$ (not necessarily a geodesic), a vector field $X$ is called parallel, or constant, along $\gamma$ if

\[ \nabla_{\frac{d}{d\tau}} X = 0. \]

In coordinates we can write

\[ \frac{d}{d\tau} = \frac{dx^i}{d\tau} \frac{\partial}{\partial x^i} \]

Using the axioms of covariant differentiation, we compute

\[ \nabla_{\frac{d}{d\tau}} X = \frac{dx^i}{d\tau} \nabla_{\frac{d}{d\tau}} \left( X^j \frac{\partial}{\partial x^j} \right) \]

\[ = \frac{dx^i}{d\tau} \nabla_{\frac{d}{d\tau}} \left( X^j \frac{\partial}{\partial x^j} \right) \]

\[ = \frac{dx^i}{d\tau} \frac{\partial}{\partial x^i} \left( X^j \frac{\partial}{\partial x^j} \right) + \frac{dx^i}{d\tau} X^j \nabla_{\frac{d}{d\tau}} \frac{\partial}{\partial x^j} \]

\[ = \frac{dX^i}{d\tau} \frac{\partial}{\partial x^i} + \frac{dx^i}{d\tau} X^j \Gamma^k_{ij} \frac{\partial}{\partial x^k} \]

\[ = \left( \frac{dX^k}{d\tau} + \frac{dx^i}{d\tau} X^j \Gamma^k_{ij} \right) \frac{\partial}{\partial x^k}. \]

The Christoffel symbols $\Gamma^k_{ij}$, the path $\gamma$, and the derivatives $\frac{dx^i}{d\tau}$ are known. Therefore the parallel transport equation is a system of $n$ first order linear differential equations in the unknowns $X^k$:

\[ \nabla_{\frac{d}{d\tau}} X = 0 \quad \text{if and only if} \quad \frac{dX^k}{d\tau} + X^j \frac{dx^i}{d\tau} \Gamma^k_{ij} = 0 \quad \text{for all } 1 < k < n. \]

This means that given a single vector $X \in T_{\gamma(0)}M$, it can be transported along the curve $\gamma$ by solving this system of equations.
2 The Riemann tensor

Given three vector fields $X$, $Y$, and $Z$, the Riemann tensor $R$ is defined as follows:

$$R(X, Y) Z \triangleq \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$  

Essentially the Riemann tensor measures the failure of commutativity of mixed second partials when applied to vector fields. Note the obvious fact that $R$ is anti-symmetric in the first two variables:

$$R(X, Y) Z = -R(Y, X) Z.$$

The Riemann tensor is also called the curvature tensor. Although at this point it is not immediately clear why, this tensor captures all of the geometric information of the space under consideration.

3 The Jacobi equation

Here we shall lay out the first geometric interpretation of the Riemann tensor: it governs variations of geodesics.

Let $p \in M$ be a point, let $\gamma(\tau)$ be a geodesic, and let $\gamma_\delta(\tau)$ be a variation consisting of a family of geodesics emanating from $p$. Again we have the directional field $\frac{d}{d\tau}$ and the variational field $\frac{d}{ds}$. We want to compute the second derivative of the variational field along the geodesic:

$$\nabla_{\frac{d}{d\tau}} \nabla_{\frac{d}{d\tau}} \frac{d}{ds}.$$  

The strategy will be to move the $\frac{d}{ds}$ to the leftmost position. First use the torsion-free axiom to make the swap

$$\nabla_{\frac{d}{ds}} \frac{d}{ds} = \nabla_{\frac{d}{ds}} \nabla_{\frac{d}{d\tau}} \frac{d}{d\tau} + \nabla_{\frac{d}{d\tau}} \left[ \frac{d}{d\tau}, \frac{d}{ds} \right].$$

However, $[d/d\tau, d/ds] = 0$, by the commutativity of partial derivatives. Now using the definition of the curvature tensor (and again using $[d/d\tau, d/ds] = 0$):

$$R \left( \frac{d}{d\tau}, \frac{d}{ds} \right) \frac{d}{d\tau} = \nabla_{\frac{d}{d\tau}} \nabla_{\frac{d}{d\tau}} \frac{d}{d\tau} - \nabla_{\frac{d}{d\tau}} \nabla_{\frac{d}{d\tau}} \frac{d}{d\tau}$$

we can make another switch:

$$\nabla_{\frac{d}{d\tau}} \nabla_{\frac{d}{d\tau}} \frac{d}{ds} = \nabla_{\frac{d}{d\tau}} \nabla_{\frac{d}{d\tau}} \frac{d}{d\tau}$$

$$= \nabla_{\frac{d}{d\tau}} \nabla_{\frac{d}{d\tau}} \frac{d}{d\tau} + R \left( \frac{d}{d\tau}, \frac{d}{d\tau} \right) \frac{d}{d\tau}.$$
However, we had assumed that the variation consists of geodesics, meaning \( \nabla \frac{d}{d\tau} = 0 \). Thus

\[
\nabla \frac{d}{d\tau} \nabla \frac{d}{d\tau} \frac{d}{ds} = R \left( \frac{d}{d\tau}, \frac{d}{ds} \right) \frac{d}{d\tau}.
\]

This is most commonly written in the following form (recalling the anti-symmetry of \( R \))

\[
\nabla \frac{d}{d\tau} \nabla \frac{d}{d\tau} \frac{d}{ds} + R \left( \frac{d}{ds}, \frac{d}{d\tau} \right) \frac{d}{d\tau} = 0.
\]

The tensor \( R(\cdot, X)X \) is known as the \textit{tidal curvature operator} in the direction \( X \).

### 4 Examples of Jacobi equations

**Example: Flat space.** In flat (Euclidean) space, the Riemann tensor is precisely zero. Thus the Jacobi equation yields

\[
\nabla \frac{d}{d\tau} \nabla \frac{d}{d\tau} \frac{d}{ds} = 0,
\]

Which means that \( \frac{d}{ds} \) is a linear field.

**Example: Positively curved space.** In a positively curved space, the tidal curvature

\[
R \left( \frac{d}{ds}, \frac{d}{d\tau} \right) \frac{d}{d\tau} \sim \alpha^2 \frac{d}{ds} + \text{other terms}
\]

is, roughly speaking, proportional to a positive multiple of the variation field. Thus the Jacobi equation yields

\[
\nabla \frac{d}{d\tau} \nabla \frac{d}{d\tau} \frac{d}{ds} + \alpha^2 \frac{d}{ds} + \text{other terms} = 0
\]

which is an equation of the form

\[
f''(\tau) + \alpha^2 f(\tau) = 0,
\]

the solution to which is \( f(\tau) = \sin(\alpha \tau) \) (initial condition is \( f(\tau) = 0 \)). Thus geodesics tend to curve in toward one another.

**Example: Negatively curved space.** In a negatively curved space, the tidal curvature

\[
R \left( \frac{d}{ds}, \frac{d}{d\tau} \right) \frac{d}{d\tau} \sim -\alpha^2 \frac{d}{ds} + \text{other terms}
\]
is, roughly speaking, proportional to a negative multiple of the variation field. Thus the Jacobi equation yields

$$\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} \frac{d}{ds} - \alpha^2 \frac{d}{ds} + \text{other terms} = 0$$

which is an equation of the form

$$f''(\tau) - \alpha^2 f(\tau) = 0,$$

the solution to which is $f(\tau) = \sinh(\alpha \tau)$ (initial condition is $f(\tau) = 0$). The function sinh is initially nearly linear, but later is nearly exponential. Thus geodesics tend to bend away from one another.

## 5 Sectional Curvature

Given two vectors $X$ and $Y$ located at a point $p$, in the infinitesimal sense they span a plane. The sectional curvature of this plane is given by

$$\sec(X, Y) = \frac{\langle R(X, Y)Y, X \rangle}{\langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2}.$$