Lecture 22 - Gauge invariance, wave equations, and variation of curves

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1 Wave equations

Consider again Maxwell’s equations
\[
\begin{align*}
\nabla \cdot \vec{B} &= 0 \quad \text{no magnetic sources} \\
\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 \quad \text{Faraday’s law} \\
\nabla \times \vec{B} - \epsilon \mu \frac{\partial \vec{E}}{\partial t} &= 4\pi \mu \vec{J} \quad \text{Ampere–Maxwell law} \\
\n\nabla \cdot \vec{E} &= \frac{4\pi}{c} \rho \quad \text{Gauss’ Law}
\end{align*}
\]

Recall the classical vector identity, valid for any vector field \( \vec{A} \):
\[
\nabla \times \nabla \times \vec{A} = \nabla (\nabla \cdot \vec{A}) - \triangle \vec{A}.
\]

Now consider the charge-free (i.e. free space) situation, in which \( \rho = 0, \vec{J} = 0 \). Taking the curl of both side of the Ampere-Maxwell equation, we get
\[
\begin{align*}
\nabla \times \nabla \times \vec{B} - \epsilon \mu \frac{\partial}{\partial t} \left( \nabla \times \vec{E} \right) &= 0 \\
\n\nabla (\nabla \cdot \vec{B}) - \triangle \vec{B} + \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} &= 0 \\
\triangle \vec{B} - \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} &= 0
\end{align*}
\]

In the second line we used Faraday’s law, and in the third line we used \( \nabla \cdot \vec{B} = 0 \). Note also that \( c^2 = \frac{1}{\epsilon \mu} \). This is the classic wave equation. Similarly, had one taken the curl of both sides of the Faraday equation to start with, then used to Ampere-Maxwell equation to simplify the result, one would get the corresponding wave equation for the \( \vec{E} \)-field:
\[
\triangle \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0.
\]
2 Gauge invariance

Electromagnetics is an example of what is known as a gauge theory. We shall describe what this means in this section.

Maxwell’s first equation $\nabla \cdot \vec{B} = 0$ implies that $\vec{B}$ is a pure curl, so that $\vec{B} = \nabla \times \vec{A}$ for some vector field $\vec{A}$. We have previously noted that there is considerable freedom in choosing $\vec{A}$, namely $\vec{A}$ can be replaced by $\vec{A} + \nabla f$ for any (differentiable) function $f$.

In the electrostatic case, $\nabla \times \vec{E} = 0$ implies that $\vec{E}$ is a pure gradient: $\vec{E} = \nabla \phi$ for some function $\phi$, called the static electric potential.

In the general case, we have $\nabla \times \vec{E} + \partial / \partial t \nabla \times \vec{A} = 0$, so that $\vec{E} + \partial \vec{A} / \partial t$ is a pure gradient. Thus there exists a function $\varphi$ so that

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = \nabla \varphi.$$  

This function $\varphi$ is called the electric pseudopotential.

Since there is some freedom in the choice of $\vec{A}$, there will be freedom in choosing $\varphi$ as well. A particular choice of $(\vec{A}, \varphi)$ is called a choice of gauge. If $f$ is any function, then replacing $(\vec{A}, \varphi)$ with $(\vec{A} + \nabla f, \varphi + \partial f / \partial t)$ does not change the equations $\vec{B} = \nabla \times \vec{A}$ or $\vec{E} + \partial \vec{A} / \partial t = \nabla \varphi$. This is known as gauge invariance.

To solve an electrodynamical problem, one usually chooses a propitious gauge (something that leads to a lot of cancelation), and then works with the more ‘primitive’ quantities $(\vec{A}, \varphi)$ instead of $\vec{E}, \vec{B}$.

3 Path variation

Let $\gamma(\tau), 0 < \tau < a$ be a path, with velocity vector $d / d\tau$. Our aim is to discover the conditions under which $\gamma(\tau)$ is a geodesic, which is to say, under what conditions it is the shortest path between its endpoints $p = \gamma(0)$ and $q = \gamma(a)$.

Recall the pathlength and the energy functionals

$${\mathcal{L}}(\gamma) = \int_0^a \sqrt{\left\langle \frac{d}{d\tau}, \frac{d}{d\tau} \right\rangle} \, d\tau$$

$${\mathcal{E}}(\gamma) = \int_0^a \left\langle \frac{d}{d\tau}, \frac{d}{d\tau} \right\rangle \, d\tau.$$ 

These are closely related, although the energy functional $\mathcal{E}$ is easier to work with since there is no square-root.
Let $\gamma_s(\tau)$ be a smoothly family of paths, parametrized by $s \in (-b, b)$, with $\gamma_0(\tau) = \gamma(\tau)$ being the original path. Assume also that each path $\gamma_s$ has the same endpoints as $\gamma$: $\gamma_s(0) = \gamma(0)$ and $\gamma_s(a) = \gamma(a)$ for all $s$. This is called a variation of $\gamma$.

There are two vector fields involved. First is $\frac{d}{d\tau}$, created by fixing $s$ and varying $\tau$. This is called the direction field. Second, it is possible to fix $\tau$ and vary $s$. This leads to the variation field $\frac{d}{ds}$.

If $\gamma$ is indeed a geodesic, then is must be the case that $\mathcal{L}(\gamma_0) \leq \mathcal{L}(\gamma_s)$, or likewise, that $\mathcal{E}(\gamma_0) \leq \mathcal{E}(\gamma_s)$. Since $\mathcal{E}(\gamma_s)$, the energy the path $\gamma_s$, can be considered a function of $s$, this means that

$$\left. \frac{d}{ds} \right|_{s=0} \mathcal{E}(\gamma_s) = 0.$$  

The key is this: *This must hold true not just for one variation, but for any conceivable variation of $\gamma$. What property of $\gamma$ could possibly lead to this?*

We compute:

$$\left. \frac{d}{ds} \right|_{s=0} \mathcal{E}(\gamma_s) = \left. \frac{d}{ds} \right|_{s=0} \int_0^a \left\langle \frac{d}{d\tau}, \frac{d}{d\tau} \right\rangle d\tau = \int_0^a d \left\langle \frac{d}{d\tau}, \frac{d}{d\tau} \right\rangle d\tau = 2 \int_0^a \left\langle \nabla_{\frac{d}{d\tau}} \frac{d}{d\tau} \right\rangle d\tau.$$  

The last line comes from the axiom that the connection $\nabla$ is compatible with the metric. Now we work some switcheroo magic. From the connection’s torsion-free axiom, we have

$$\nabla_{\frac{d}{d\tau}} \frac{d}{d\tau} = \nabla_{\frac{d}{d\tau}} \frac{d}{d\tau} + \left[ \frac{d}{d\tau}, \frac{d}{d\tau} \right]$$

But the bracket in this case actually vanishes! (why?) Thus

$$\nabla_{\frac{d}{d\tau}} \frac{d}{d\tau} = \nabla_{\frac{d}{d\tau}} \frac{d}{d\tau},$$

and we can continue our computation

$$\left. \frac{d}{ds} \right|_{s=0} \mathcal{E}(\gamma_s) = 2 \int_0^a \left\langle \nabla_{\frac{d}{d\tau}} \frac{d}{d\tau}, \frac{d}{d\tau} \right\rangle d\tau = 2 \int_0^a \frac{d}{d\tau} \left\langle \frac{d}{d\tau}, \frac{d}{d\tau} \right\rangle d\tau - 2 \int_0^a \left\langle \frac{d}{d\tau}, \nabla_{\frac{d}{d\tau}} \frac{d}{d\tau} \right\rangle d\tau.$$  

The first term is a total derivative; by the fundamental theorem of calculus, we have

$$\left. \frac{d}{ds} \right|_{s=0} \mathcal{E}(\gamma_s) = 2 \left. \left\langle \frac{d}{d\tau}, \frac{d}{d\tau} \right\rangle \right|_{\tau=0} - 2 \int_0^a \left\langle \frac{d}{d\tau}, \nabla_{\frac{d}{d\tau}} \frac{d}{d\tau} \right\rangle d\tau = -2 \int_0^a \left\langle \frac{d}{d\tau}, \nabla_{\frac{d}{d\tau}} \frac{d}{d\tau} \right\rangle d\tau.$$
If \( \gamma \) is indeed the minimizing path between its endpoints, we must therefore have
\[
0 = -2 \int_0^a \left\langle \frac{d}{ds}, \nabla_{\frac{d}{d\tau}} \frac{d}{d\tau} \right\rangle d\tau
\]
for ANY variation field \( \frac{d}{ds} \). The only way this is possible is if \( \nabla_{\frac{d}{d\tau}} \frac{d}{d\tau} = 0 \).

Therefore we can write down the geodesic equation:
\[
\gamma \text{ is a geodesic} \iff \nabla_{\frac{d}{d\tau}} \frac{d}{d\tau} = 0
\]