1 Euclidean analytic geometry

Euclidean \( n \)-space has no natural origin and so is not naturally a vector space, despite how we are often taught to regard it. It makes no sense, intrinsically, to speak of a “position vector.”

Nevertheless, when working in 3-space say, we often choose an origin and post \( x \)-, \( y \)-, and \( z \)-coordinate axes, and regard it as a vector space with basis \((1,0,0)\), \((0,1,0)\), and \((0,0,1)\). It is important to remember that Euclidean space is a purely geometric object and does not come equipped with axes. Supplying space with axes is a purely arbitrary construction.

Euclidean space does however come with notions of angles, distances, and lines. Once an orthonormal coordinate system is chosen, we may regard the geometric object of Euclidean \( n \)-space as the algebraic object \( \mathbb{R}^n \), which does happen to be a vector space, and is easier to work with on paper.

1.1 Paths, pathlength, and path integrals

Consider Euclidean \( n \)-space, and choose an orthonormal coordinate system \((x^1, \ldots, x^n)\), thereby identifying it with \( \mathbb{R}^n \).

Consider a path \( \Gamma(t) = (x^1(t), \ldots, x^n(t)) \) through \( n \)-space. The path’s velocity is

\[
\frac{d\Gamma}{dt}(t) = \left( \frac{dx^1}{dt}, \ldots, \frac{dx^n}{dt} \right)_{(x^1(t), \ldots, x^n(t))}.
\]
It's speed is the velocity vector's length:

\[
\left| \frac{d\Gamma}{dt} \right| = \sqrt{\langle \frac{d\Gamma}{dt}, \frac{d\Gamma}{dt} \rangle} = \sqrt{\left( \frac{dx^1}{dt} \right)^2 + \ldots + \left( \frac{dx^n}{dt} \right)^2}.
\]

If \( s \) is the arclength, we have \( ds = |d\Gamma| = |d\Gamma/dt| dt \). To find the length of the path \( \Gamma(t) \) between \( t = t_1 \) and \( t = t_2 \), you integrate the path’s arclength:

\[
\int_{t_1}^{t_2} ds = \int_{t_1}^{t_2} \left| \frac{d\Gamma}{dt} \right| dt.
\]

Given a function \( f(x^1, \ldots, x^n) \) defined on \( \mathbb{R}^n \), it is possible to integrate \( f \) along the path \( \Gamma \):

\[
\int_{\Gamma} f ds = \int_{t_1}^{t_2} f(x^1(t), \ldots, x^n(t)) \left| \frac{d\Gamma}{dt} \right| dt.
\]

Finally, it is possible to take the derivative of \( f \) along the path \( \Gamma \). Using the chain rule, we get

\[
\frac{df}{dt} = \frac{dx^1}{dt} \frac{\partial f}{\partial x^1} + \frac{dx^2}{dt} \frac{\partial f}{\partial x^2} + \ldots + \frac{dx^n}{dt} \frac{\partial f}{\partial x^n}.
\]

This indicates that we can write

\[
\frac{d}{dt} = \frac{dx^1}{dt} \frac{\partial}{\partial x^1} + \frac{dx^2}{dt} \frac{\partial}{\partial x^2} + \ldots + \frac{dz^n}{dt} \frac{\partial}{\partial x^n}.
\]

2 Minkowski analytic geometry

2.1 Minkowski’s Pythagorean theorem

Consider the spaceships from the previous lecture. From our perspective, the crewmember traveled \( \Delta x \) meters between when the light left the flashlight and when it struck the ceiling, and did so in \( \Delta t \) seconds. The crewmember’s own experience is that no distance was traveled, but that \( \Delta \tau \) seconds passed. Using the time-dilation equation \( \Delta t = \gamma \Delta \tau \), let’s compute

\[
(\Delta t)^2 - \frac{1}{c^2}(\Delta x)^2 = (\Delta t)^2 \left( 1 - \frac{v^2}{c^2} \right) = \gamma^2(\Delta \tau)^2 \gamma_v^{-2} = (\Delta \tau)^2.
\]
Graphically, $\Delta t$ and $\Delta x$ form the legs of a right triangle with hypotenuse $\Delta \tau$, so this gives a new pythagorean theorem. If $p$ and $\bar{p}$ are points in space-time (“events”), then the “distance” (rather, proper time) between them is given by

$$p = (x^0, x^1, x^2, x^3)$$

$$\bar{p} = (\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3)$$

$$|\bar{p}p|^2 = (\Delta x^0)^2 - \frac{1}{c^2} (\Delta x^1)^2 - \frac{1}{c^2} (\Delta x^2)^2 - \frac{1}{c^2} (\Delta x^3)^2$$

$$= (x^0 - \bar{x}^0)^2 - \frac{1}{c^2} (x^1 - \bar{x}^1)^2 - \frac{1}{c^2} (x^2 - \bar{x}^2)^2 - \frac{1}{c^2} (x^3 - \bar{x}^3)^2.$$  

Four-space that obeys this version of the Pythagorean theorem is called Minkowski space.

The distinction between $\mathbb{R}^4$ and $\mathbb{R}^{1,3}$. After choosing a coordinate system for space-time (a.k.a. inertial frame of reference) with coordinates $(x^0, x^1, x^2, x^3)$, space-time becomes identified with $\mathbb{R}^4$. But notice that the time and space dimensions are treated differently due to our new Pythagorean theorem: the square of the time-dimension has a coefficient of $+1$, whereas the squared space-dimensions have coefficients $-1/c^2$ (giving these terms, by the way, units of time-squared). Because the geometry Minkowski space is so radically different from the geometry of Euclidean space, we usually call it $\mathbb{R}^{1,3}$ instead of $\mathbb{R}^4$, or sometimes “1+3 dimensional space” rather than “4 dimensional space.”

Notice that proper time (i.e. distance in 1+3-space) can either be real and positive, zero, or imaginary. Respectively, these are called “time-like”, “light-like” or “null”, and “space-like” distances.

### 2.2 Paths in Minkowski space

Given an arbitrary path

$$\Gamma(t) = (x^0(t), x^1(t), x^2(t), x^3(t))$$

through space-time, it’s four-velocity is

$$\frac{d\Gamma}{dt} = \left(\frac{dx^0}{dt}, \frac{dx^1}{dt}, \frac{dx^2}{dt}, \frac{dx^3}{dt}\right).$$

The meaning of this vector is not the same as for paths through Euclidean space: indeed, the 4-velocity is the path’s velocity through proper time. Note that the path’s speed through proper time is

$$\left|\frac{d\Gamma}{dt}\right| = \sqrt{\left(\frac{dx^0}{dt}\right)^2 - \frac{1}{c^2} \left(\frac{dx^1}{dt}\right)^2 - \frac{1}{c^2} \left(\frac{dx^2}{dt}\right)^2 - \frac{1}{c^2} \left(\frac{dx^3}{dt}\right)^2}.$$
This could be well imaginary! At points where the speed is positive the path is called “time-like”, at points where the speed is 0 the path is called “light-like” or “null”, and at points where the speed is imaginary the path is called “space-like”.

If $s$ is the arclength parameter (perhaps the term “arctime parameter” is better) along the path $\Gamma$ and $ds$ the arclength element, the infinitesimal Pythagorean theorem for Minkowski space gives

$$(ds)^2 = (dx^0)^2 - \frac{1}{c^2} (dx^1)^2 - \frac{1}{c^2} (dx^2)^2 - \frac{1}{c^2} (dx^3)^2.$$ 

Thus the Minkowski arclength is

$$\int_{\Gamma} ds = \int_\Gamma \sqrt{(dx^0)^2 - \frac{1}{c^2} (dx^1)^2 - \frac{1}{c^2} (dx^2)^2 - \frac{1}{c^2} (dx^3)^2} dt$$

This is the total time measured along the path from $\Gamma(t_1)$ to $\Gamma(t_2)$.

Path of a physical particle In the case where the path $\Gamma(t)$ represents the motion of a physical particle through space-time, we impose a physicality condition: that the rate of passage of proper time measured by the particle is unity. Namely, we impose the condition:

$$\left| \frac{d\Gamma}{dt}(t) \right| = 1.$$ 

3 Formulas

Given a frame $(x^0, x^1, x^2, x^3)$ for $\mathbb{R}^{1,3}$ and a line segment of length $\Delta s$, we have the following relationship between proper time and space-time measurements:

$$(\Delta s)^2 = (\Delta x^0)^2 - \frac{1}{c^2} (\Delta x^1)^2 - \frac{1}{c^2} (\Delta x^2)^2 - \frac{1}{c^2} (\Delta x^3)^2.$$