Lecture 17 - Relation between the classical vector operations and the $d$-operator. Also 4-velocity and 3-velocity, and 4-momentum.

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1 Classical Vector Analysis

In classical 3-dimensional vector analysis, we always work with standard orthonormal coordinates $(x, y, z)$ (never polar or cylindrical coordinates) and we consider a vector at a point $p$ to be an assignment of a list of 3 numbers to $p$. A vector field is an assignment of a list of 3 numbers to each point of a region of space. Typically vector fields are denoted in capitals: $X, Y, W$, etc.

Two algebraic operations

Let $X = (a, b, c)$ and $Y = (\alpha, \beta, \gamma)$ be two vectors based at a point $p$. We define their inner product and their cross product to be

$$\langle X, Y \rangle = a\alpha + b\beta + c\gamma$$

$$= |X||Y| \cos \theta$$

$$X \times Y = (b\gamma - c\beta, c\alpha - a\gamma, a\beta - b\alpha)$$

$$= \hat{n}|X||Y| \sin \theta,$$

where $\theta$ is the angle between $x$ and $Y$, and $\hat{n}$ is the unique unit vector normal to both $X$ and $Y$ obtained using the right-hand rule.

Three analytic operations

Let $X = (a, b, c)$ be a vector, and let $f$ be a function. We define the gradient of $f$ to be the vector field

$$\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right).$$

It is interpreted to be the vector whose direction at each point is that of maximum increase of the function $f$, and whose length indicates the rate of that increase.
We define the divergence of the vector field \( X \) to be the function
\[
\nabla \cdot X = \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}.
\]
The divergence of a vector field at a point is interpreted to be its tenancy to rarify or accumulate at that point.

We define the curl of the vector field \( X \) to be the vector field
\[
\nabla \times X = \left( \frac{\partial c}{\partial y} - \frac{\partial b}{\partial z}, \frac{\partial a}{\partial z} - \frac{\partial c}{\partial x}, \frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} \right).
\]
The curl of a vector field at a point is interpreted to be the tenancy of the vector field to rotate about a point, where the direction of \( \nabla \times X \) is the axis of rotation (according to the right-hand rule) and the length of \( \nabla \times X \) is the magnitude of the rotation.

Two identities

The following two identities are theorems of classical analysis:
\[
\nabla \times (\nabla f) = 0
\]
\[
\nabla \cdot (\nabla \times X) = 0.
\]

2 The \( d \)-operator and the classical vector operations

The classical vector operations all have equivalents in the language of forms. The three analytic operations can be recovered using the \( d \)-operator in various ways.

The dot product

This is just the inner product: if \( X = X^i \frac{\partial}{\partial x^i} \) and \( Y = Y^i \frac{\partial}{\partial x^i} \), then
\[
\langle X, Y \rangle = g_{ij} X^i Y^j.
\]
Note that if \( g_{ij} = \delta_{ij} \) is the Euclidean inner product, then this is the classical dot product.

The cross product

Let \( X = adx + bdy + cdz \) and \( Y = \alpha dx + \beta dy + \gamma dz \) be 1-forms. Then \( X \wedge Y \) is the 2-form given by
\[
X \wedge Y = (adx + bdy + cdz) \wedge (\alpha dx + \beta dy + \gamma dz)
= (b\gamma - c\beta) dy \wedge dz + (a\gamma - c\alpha) dz \wedge dx + (a\beta - b\alpha) dx \wedge dy.
\]

The gradient

Let \( f \) be a 0-form (a function). Then
\[
df = \frac{df}{dx} dx + \frac{df}{dy} dy + \frac{df}{dz} dz.
\]
The curl

Let \( \omega = adx + bdy + xdz \) be a 1-form. Then by the definition of \( d\omega \) we have

\[
d\omega = d(a) \wedge dx + d(b) \wedge dy + d(c) \wedge dz
\]=
\[
\begin{align*}
\frac{da}{dx} dx \wedge dx + \frac{da}{dy} dy \wedge dx + \frac{da}{dz} dz \wedge dx
&+ \frac{db}{dx} dx \wedge dy + \frac{db}{dy} dy \wedge dy + \frac{db}{dz} dz \wedge dy \\
&+ \frac{dc}{dx} dx \wedge dz + \frac{dc}{dy} dy \wedge dz + \frac{dc}{dz} dz \wedge dz
\end{align*}
\]  
\[=
\left( \frac{da}{dx} - \frac{db}{dy} \right) dy \wedge dz + \left( \frac{da}{dz} - \frac{dc}{dx} \right) dz \wedge dx + \left( \frac{db}{dx} - \frac{da}{dy} \right) dx \wedge dy.
\]

The divergence

Let \( \omega = ady \wedge dz + bdz \wedge dx + cdx \wedge dy \) be a 2-form. We compute \( d\omega \):

\[
d\omega = d(a) \wedge dy \wedge dz + d(b) \wedge dz \wedge dx + d(c) \wedge dx \wedge dy
\]=
\[
\begin{align*}
\frac{da}{dx} dx \wedge dy \wedge dz + \frac{da}{dy} dy \wedge dy \wedge dz + \frac{da}{dz} dz \wedge dy \wedge dz
&+ \frac{db}{dx} dx \wedge dz \wedge dx + \frac{db}{dy} dy \wedge dz \wedge dx + \frac{db}{dz} dz \wedge dz \wedge dx \\
&+ \frac{dc}{dx} dx \wedge dx \wedge dy + \frac{dc}{dy} dy \wedge dx \wedge dy + \frac{dc}{dz} dz \wedge dx \wedge dy
\end{align*}
\]  
\[=
\left( \frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz} \right) dx \wedge dy \wedge dz.
\]

Two identities in one

In our new notation, the two vector identities \( \nabla \times \nabla f = 0 \) and \( \nabla \cdot (\nabla \times \mathbf{X}) = 0 \) coalesce into the single fact that \( dd = 0 \).

Given a function \( f \), the 1-form \( df \) is equivalent to the classical gradient. Given a 1-form \( \omega \), the 2-form \( d\omega \) is equivalent to the classical curl. This means \( ddf \) is the classical analog of the curl of the gradient, and the fact that \( ddf = 0 \) is equivalent to the classical theorem \( \nabla \times \nabla f = 0 \).

Given a 2-form \( \eta \), the 3-form \( d\eta \) is equivalent to the classical divergence. If \( \omega \) is a 1-form, then \( d\omega \) is equivalent to the divergence of the curl, and the fact that \( d\omega = 0 \) is equivalent to the classical theorem \( \nabla \cdot (\nabla \times \mathbf{X}) = 0 \).

3 A comment on notation

In the context of Minkowski space, indices \( i, j, k \), etc will sum from 0 to 3. Indices \( a, b, c \), etc will sum from 1 to 3 (ie, only over the space-dimensions, not time). For instance, in
coordinates a 4-vector $v$ will be denoted
\[ v = v^i \frac{\partial}{\partial x^i} = v^0 \frac{\partial}{\partial x^0} + v^1 \frac{\partial}{\partial x^1} + v^2 \frac{\partial}{\partial x^2} + v^3 \frac{\partial}{\partial x^3}, \]
and a 3-vector $\vec{v}$ will be denoted
\[ \vec{v} = \vec{v}^a \frac{\partial}{\partial x^a} = \vec{v}^1 \frac{\partial}{\partial x^1} + \vec{v}^2 \frac{\partial}{\partial x^2} + \vec{v}^3 \frac{\partial}{\partial x^3}. \]

4 4-velocity and 3-velocity

Let $\gamma(\tau)$ be a path through space-time, parameterized by $\tau$. Typically $\gamma$ represents a particle’s worldline, its path through space-time. Given coordinates $\{x^i\}$, the velocity vector of $\gamma$ is
\[ v = \frac{d}{d\tau} = \frac{dx^i}{d\tau} \frac{\partial}{\partial x^i}. \]
If the coordinates $\{x^i\}$ constitute an inertial reference frame, we can define the particle’s 3-velocity:
\[ \vec{v} = \frac{dx^a}{dx^0} \frac{\partial}{\partial x^a} = \frac{dx^1}{dx^0} \frac{\partial}{\partial x^1} + \frac{dx^2}{dx^0} \frac{\partial}{\partial x^2} + \frac{dx^3}{dx^0} \frac{\partial}{\partial x^3} = \frac{dx^1/d\tau}{dx^0/d\tau} \frac{\partial}{\partial x^1} + \frac{dx^2/d\tau}{dx^0/d\tau} \frac{\partial}{\partial x^2} + \frac{dx^3/d\tau}{dx^0/d\tau} \frac{\partial}{\partial x^3} \]
This indicates the rate of change of the particle’s position with respect to coordinate time.

Recall the Minkowski metric:
\[ g_{11} = 1, \quad g_{22} = -\frac{1}{c^2}, \quad g_{22} = -\frac{1}{c^2}, \quad g_{33} = -\frac{1}{c^2} \]
Given a 4-vector $v = v^i \frac{\partial}{\partial x^i}$, we define its norm-square is
\[ |v|^2 = g_{ij} v^i v^j = v^0 v^0 - \frac{1}{c^2} v^1 v^1 - \frac{1}{c^2} v^2 v^2 - \frac{1}{c^2} v^3 v^3 \]
(recall that this can be positive, negative, or zero).

We will indicate 3-vectors by using an over-arrow: $\vec{v}$ means $\vec{v}$ is a 3-vector. If $\vec{v}$ is a 3-tensor, we define its norm-square to be the Euclidean norm-square:
\[ |\vec{v}|^2 = \delta_{ab} v^a v^b = v^1 v^1 + \ldots + v^n v^n. \]
We impose the following physicality condition on timelike paths:
• If $\gamma(\tau)$ represents the path of a massive particle through space-time, its velocity vector $v = \frac{d}{d\tau}$ is time-like and $|v|^2 = +1$.

Using the physicality condition, we can compute the component $\frac{dx^0}{d\tau}$ for time-like paths:

$$1 = \left| \frac{d}{d\tau} \right|^2 = \left( \frac{dx^0}{d\tau} \right)^2 - \frac{1}{c^2} \left( \frac{dx^1}{d\tau} \right)^2 - \frac{1}{c^2} \left( \frac{dx^2}{d\tau} \right)^2 - \frac{1}{c^2} \left( \frac{dx^3}{d\tau} \right)^2$$

$$\frac{1}{\left( \frac{dx^0}{d\tau} \right)^2} = 1 - \frac{1}{c^2} \left( \frac{dx^1}{dx^0} \right)^2 - \frac{1}{c^2} \left( \frac{dx^2}{dx^0} \right)^2 - \frac{1}{c^2} \left( \frac{dx^3}{dx^0} \right)^2$$

$$\left( \frac{dx^0}{d\tau} \right)^2 = \frac{1}{\left( \frac{dx^0}{d\tau} \right)^2 - \left| \vec{v} \right|^2}$$

$$\frac{dx^0}{d\tau} = \gamma |v|$$

Using this, we have

$$v = v^i \frac{\partial}{\partial x^i}$$

$$\vec{v} = \vec{v}^a \frac{\partial}{\partial x^a}$$

where $v^0 = \gamma$, $v^a = \gamma \vec{v}^a$. A shorthand way to write this is to use “classical” vector notation:

$$\vec{v} \triangleq \left( \frac{dx^1}{dx^0}, \ldots, \frac{dx^n}{dx^0} \right) = (v^1, \ldots, v^n)$$

$$v \triangleq \left( \frac{dx^0}{d\tau}, \frac{dx^1}{d\tau}, \ldots, \frac{dx^n}{d\tau} \right)$$

$$= (v^0, v^1, \ldots, v^n) = (\gamma, \gamma v^1, \ldots, \gamma v^n)$$

$$= (\gamma, \gamma \vec{v}).$$

5 Momentum

If $v = v^i = v^i \frac{\partial}{\partial x^i}$ is a velocity vector, we define the corresponding momentum (or conjugate momentum) covector to be

$$p = -mc^2 v_s,$$
or, in components,

\[ p_i = -mc^2 g_{ij} v^j. \]

Recall the Minkowski metric

\[ g_{00} = 1 \quad g_{11} = -\frac{1}{c^2} \quad g_{22} = -\frac{1}{c^2} \quad g_{33} = -\frac{1}{c^2}. \]

Then we have

\[ v = v^0 \frac{\partial}{\partial x^0} + v^1 \frac{\partial}{\partial x^1} + v^2 \frac{\partial}{\partial x^2} + v^3 \frac{\partial}{\partial x^3} \]
\[ p = -mc^2 v^0 dx^0 + mv^1 dx^1 + mv^2 dx^2 + mv^3 dx^3. \]

Using vector notation

\[ v = (\gamma, \gamma \vec{v}) \frac{\partial}{\partial x^i}, \]
\[ p = -mc^2 \left( \gamma, \frac{1}{c^2} \gamma \vec{v} \right) dx^i \]
\[ = \left( -mc^2 \gamma, m \gamma \vec{v} \right) dx^i = \left( -mc^2 \gamma, \vec{p} \right). \]

Note the space-components:

\[ \vec{p} = m \gamma \vec{v}. \]

This closely resembles the classical 3-momentum \( \vec{p} = m \vec{v}. \)