The Hopf Fibration and the Berger Spheres

Due —

Introduction \( \mathbb{R}^4 \) is the set of ordered quadruples of real numbers \((x_1, x_2, x_3, x_4)\), along with the Euclidean distance function:

\[
\text{dist} \left( (x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \right) = (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 + (x_4 - y_4)^2.
\]

One can identify \( \mathbb{R}^4 \) with \( \mathbb{C}^2 \), the set of ordered pairs or complex numbers: a point \((z^1, z^2) = (x^1 + iy^1, x^2 + iy^2) \in \mathbb{C}^2\) can be identified with the point \((x^1, y^1, x^2, y^2) \in \mathbb{R}^4\).

If \((z^1, z^2), (w^1, w^2)\) are two points in \( \mathbb{C}^2 \), the distance between them is

\[
\text{dist} \left( (z^1, z^2), (w^1, w^2) \right) = |z^1 - w^1|^2 + |z^2 - w^2|^2.
\]

Recall that if \( z \in \mathbb{C} \) then \(|z|^2 = z \overline{z}\).

Let \( S^3 \subset \mathbb{R}^4 \) denote the unit 3-sphere, defined to be

\[
S^3 \triangleq \left\{ (x^1, x^2, x^3, x^4) \in \mathbb{R}^4 \text{ s.t. } (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1 \right\} = \left\{ (z^1, z^2) \in \mathbb{C}^2 \text{ s.t. } |z^1|^2 + |z^2|^2 = 1 \right\}.
\]

1 The Hopf action

**Problem 1** If \( p, q \in \mathbb{C}^2 \), the distance \( \text{dist}(p, q) \) can be calculated in the \( \mathbb{C}^2 \) sense or the \( \mathbb{R}^4 \) sense. Prove that the distance is the same regardless of which distance function is used.

**Def** Given any \( \theta \in \mathbb{R} \), let \( \psi_\theta : \mathbb{C}^2 \to \mathbb{C}^2 \) be the map

\[
\psi_\theta(z^1, z^2) = (e^{i\theta} z^1, e^{i\theta} z^2).
\]

Note that \( \psi_\theta \) is the identity map if and only if \( \theta \) is a multiple of \( 2\pi \).

**Problem 2** Given any \( \theta \in \mathbb{R} \), prove that \( \psi_\theta \) is an isometry that fixes the origin \((0, 0) \in \mathbb{C}^2\).

Prove that, unless \( \theta \) is a multiple of \( 2\pi \), then \((0, 0)\) is the only fixed point of \( \psi_\theta \). Finally, prove that if \((z^1, z^2) \in S^3\) then also \( \psi_\theta(z^1, z^2) \in S^3\).

**Remark** Thus \( \psi_\theta : S^3 \to S^3 \) is an isometric action: this is known as the Hopf action.
Problem 3) If \( p = (z_1, z_2) \in S^3 \), the orbit of \( p \) under the Hopf action is defined to be the set of all \( \psi_\theta(p) \) as \( \theta \) varies. Prove that the orbit of any point \( p \in S^3 \) is a circle of radius 1.

Remark Each orbit of \( \psi_\theta \) is a circle, and of course each point of \( S^3 \) lies in an orbit. Thus the union of the orbits (each a copy of \( S^1 \)) comprises \( S^3 \). One says that \( S^3 \) is fibered by \( S^1 \); one calls the \( S^1 \) orbits the fibers. The fibration of \( S^3 \) by copies of \( S^1 \) is called the Hopf fibration.

Problem 4) Considering \( S^3 \subset \mathbb{R}^4 \) (instead of \( S^3 \subset \mathbb{C}^2 \)), prove that
\[
\psi_\theta(x^1, x^2, x^3, x^4) = (x^1 \cos \theta - x^2 \sin \theta, x^1 \sin \theta + x^2 \cos \theta, x^3 \cos \theta - x^4 \sin \theta, x^3 \sin \theta + x^4 \cos \theta).
\]
The Hopf action \( \psi_\theta \) produces an action field, which is just the velocity field of the rotation. Letting \( \frac{d}{d\theta} \) denote the action field, compute \( \frac{d}{d\theta} \) in terms of the coordinate fields \( \left\{ \frac{\partial}{\partial x^i} \right\} \).

2 The Hopf map

Def The Hopf map \( \Psi : S^3 \to S^2 \) is a continuous map defined as follows. Regard \( S^2 \) to be \( \mathbb{C} \cup \{\infty\} \) (via stereographic projection). If \( p \in S^3 \) is an arbitrary point, then \( p = (z^1, z^2) \) with \( |z^1|^2 + |z^2|^2 = 1 \). Define
\[
\Psi(p) \in \mathbb{C} \cup \{\infty\}
\]
\[
\Psi \left( (z^1, z^2) \right) = \frac{z^1}{z^2}.
\]
In contrast to the lower dimensional situation, there are NO (topologically nontrivial) continuous maps from \( S^2 \) to \( S^1 \).

Problem 5) Prove that \( \Psi : S^3 \to S^2 \) is onto. Which point in \( S^3 \) maps to the ‘point at infinity’ on \( S^2 \)?

Problem 6) Prove that \( p, q \in S^3 \) belong to the same Hopf fiber if and only if \( \Psi(p) = \Psi(q) \).

3 The Berger spheres

Remark The Hopf map \( S^3 \to S^2 \) is an example of a submersion: a map from a higher dimensional space into (in this case onto) a lower dimensional space. As we have seen, \( \Psi \) takes the 3-sphere and collapses the Hopf circles (1-dimensional objects) to points. What is left is a 2-sphere (a 2-dimensional object). The purpose of this section is to see the process occurring dynamically: we will construct a family of metrics that shrinks the Hopf circles to points.
Problem 7) Let $X$, $Y$, and $Z$ be vector fields given by

\[
X = \frac{d}{d\theta} \quad = \quad -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2} - x^4 \frac{\partial}{\partial x^3} + x^3 \frac{\partial}{\partial x^4} \\
Y = -x^3 \frac{\partial}{\partial x^1} + x^4 \frac{\partial}{\partial x^2} + x^1 \frac{\partial}{\partial x^3} - x^2 \frac{\partial}{\partial x^4} \\
Z = x^4 \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^3} - x^1 \frac{\partial}{\partial x^4}.
\]

Prove that $X$, $Y$, and $Z$ are all tangent to $S^3$. Also prove that $|X|^2 = |Y|^2 = |Z|^2 = 1$, and that $X$, $Y$, and $Z$ are mutually orthogonal.

Problem 8) Prove that

\[
[X, Y] = 2Z \\
[Y, Z] = 2X \\
[Z, X] = 2Y.
\]

This also proves that $X$, $Y$, $Z$ cannot be considered coordinate fields.

Def We shall define the Berger metric on $S^3$ as follows. Let $\eta^1 \triangleq X_\alpha$, $\eta^2 \triangleq Y_\alpha$, and $\eta^3 \triangleq Z_\alpha$. Given $\alpha \in \mathbb{R}$, let

\[
g_\alpha = \alpha^2 \eta^1 \otimes \eta^1 + \eta^2 \otimes \eta^2 + \eta^3 \otimes \eta^3.
\]

If $\alpha = 1$, then this is precisely the metric that $S^3$ inherits from the ambient space $\mathbb{R}^4$.

Problem 9) Each of the metrics $g_\alpha$ has an associated covariant derivative $\nabla^\alpha$. Find

\[
\nabla^\alpha Y \quad \nabla^\alpha Z \quad \nabla^\alpha X.
\]

(It is best to use the Koszul formula directly). Note that the values of $\nabla^\alpha Y$, $\nabla^\alpha Z$, $\nabla^\alpha X$ are now automatic.

Problem 10) Compute the sectional curvatures

\[
\sec(X, Y) \quad \sec(Y, Z) \quad \sec(X, Z).
\]

As measured in the metric $g_\alpha$, the Hopf fibers are circles of radius $\alpha$. As $\alpha \to 0$ and the fibers contract to points, the sectional curvatures remain bounded. This is a process known as collapse with bounded curvature.