Part I

1) (7 pts) Solve for $x$: \( \frac{dx}{dt} = tx - \frac{t}{x} \), \( x(0) = 2 \).

2) (8 pts) Find the general solution: \( \frac{dw}{dx} = \frac{2x - w}{x + 6y} \).
3) (15 pts) Solution with a concentration of 0.1 lbs of salt per gallon pours into a tank at a rate of \( \frac{2}{t+1} \) gallons per minute. Also, well-mixed solution leaves the tank at the same rate. How much salt is in the tank after 1 minute, if initially the tank contains 1 gallon of water mixed with 0.9 lb of salt?
4) The equation for a particle caught in the gravitational field of a body of mass $M$ is

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2},$$

where $G \approx 6.67 \times 10^{-11} m^3 kg^{-1} s^{-2}$ and $r$ is the distance to the center of mass of the body. A particle of dust is caught in the gravitational field of a small, spherically shaped asteroid of mass $\frac{16}{6.67} \times 10^{11} kg$ and radius 100m.

a) (10 pts) Use the methods of differential equations to find $\frac{dr}{dt}$.

b) (10 pts) Initially the dust particle is motionless relative to the asteroid and 300m from its surface. With what speed will it strike the asteroid’s surface?
Part II

5) Consider the differential equation

\[ x'''' + 2x''' + 2x'' = 4 - 12t \]

a) (10 pts) Find the complimentary solution

b) (15 pts) Find the general solution.

c) (5 pts) Find the solution, given \( x(0) = 0, \ x'(0) = 0, \ x''(0) = 8, \ x'''(0) = -6. \)
(There’s an easy way and a hard way.)
Part III

6) (5 pts) Compute $e^{tA}$, where
$$A = \begin{pmatrix} 0 & -1 & 6 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}.$$

7) Consider the equation
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ t \end{pmatrix}.$$ 

a) (5 pts) If you use the method of undetermined coefficients, what would your ‘guess’ for $x_p$ be?

b) (15 pts) Find $x_p$. 

Part IV

8) Consider the system

\[
\begin{pmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{pmatrix}' =
\begin{pmatrix}
    1 & 0 & 0 & 1 \\
    0 & -1 & -2 & 0 \\
    0 & 2 & -1 & 0 \\
    0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{pmatrix}
\]

a) (10 pts) Find the eigenvalues of the matrix.

b) (20 pts) Find the system’s general solution.
c) (10 pts) Write down the fundamental matrix $\Phi(t)$, and compute $\Phi(0)^{-1}$
HINT: This particular $\Phi(0)$ should have some special properties. Before trying to compute $\Phi(0)^{-1}$, see what you get when you multiply $\Phi(0) \cdot \Phi(0)^T$.

d) (5 pts) Given $x(0) = \begin{pmatrix} -1 & 1 & 0 & 2 \end{pmatrix}^T$, find $x(t)$.