Chapter 5 Review/Extra Credit Problems

Remember, no credit will be given for answers without justification.
Instructions: Do #1, either #2 or #3, and either #4 or #5.

1) Let

\[ A = \begin{pmatrix} 3 & 0 & -1 \\ -1 & 2 & 2 \\ 1 & 0 & 1 \end{pmatrix} \]

a) (5 pts) Show that \( A \) can be written as \( A = B + C \), where \( B \) is the diagonal matrix with 2’s along the diagonal, and \( C \) is a nilpotent matrix.

b) (5 pts) Use part (a) to compute \( e^{A} \).

c) (5 pts) Compute \( e^{tA} \).

d) (5 pts) Solve the initial value problem

\[
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & -1 \\ -1 & 2 & 2 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.
\]

In the following two problems, you will consider a system \( \mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f} \). Perform the following:

a) (10 pts) Find the complimentary solution

b) (5 pts) What is your ‘guess’ for the particular solution, \( \mathbf{x}_p \)?

Special instructions: Be extremely specific. Do not write \( \mathbf{x}_p = t\mathbf{a} + \mathbf{b} \) for example, but instead write \( \mathbf{x}_p = t \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \), or maybe even \( \mathbf{x}_p = \begin{pmatrix} ta_1 + b_1 \\ ta_2 + b_2 \end{pmatrix} \).

c) (10 pts) Plug in your ‘guess’ for \( \mathbf{x}_p \). You will get a system of 4 equation and 4 unknowns. Write this system in augmented matrix form.

d) (5 pts) Find \( \mathbf{x}_p \).

2) \( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 6 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} \)

3) \( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -1 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} t \\ 1 + t \end{pmatrix} \)
In the following two problems, you will consider a system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Answer the following questions:

a) (10 pts) Find all the generalized eigenvectors of $\mathbf{A}$.

b) (5 pts) Find the general solution.

c) (10 pts) Find $\Phi(t)$ and compute $\Phi(0)^{-1}$.

d) (5 pts) Find the solution, given $\mathbf{x}(0) = (-1 \ 1 \ 0 \ 1)^T$.

4) 

\[
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}' = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}
\]

5) 

\[
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}' = \begin{pmatrix} -1 & -1 & -1 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}
\]