1 Lecture 12 - Areas between graphs, and volumes of rotation

1.1 Areas between graphs

The area under the graph of $f(x)$, between $x = a$ and $x = b$ is

$$\int_a^b f(x) \, dx.$$ 

Now consider two functions $f(x)$ and $g(x)$ which intersect at $x = x_0$ and $x = x_1$. The area under $f(x)$ between $x_0$ and $x_1$ is $\int_{x_0}^{x_1} f(x) \, dx$, and the area under $g(x)$ between $x_0$ and $x_1$ is $\int_{x_0}^{x_1} g(x) \, dx$. To get the area between the graphs, you subtract:

$$\int_{x_0}^{x_1} (f(x) - g(x)) \, dx \quad \text{or} \quad \int_{x_0}^{x_1} (g(x) - f(x)) \, dx,$$

depending on which one has the upper graph and which one has the lower graph.

Example 1 Find the area bounded between the graphs of $f(x) = 3x + 4$ and $g(x) = x^2$.

Solution First we find the points of intersection by setting the functions equal to each other:

$$f(x) = g(x)$$
$$3x + 4 = x^2$$
$$0 = x^2 - 3x - 4$$
$$0 = (x - 4)(x + 1).$$

Thus the points of intersection are $x = -1$ and $x = 4$. It can easily be seen from the graphs that $f(x) = 3x - 4$ is the upper function. Thus the bounded
area is
\[ \inf_{-1}^{4} (3x + 4 - x^2) \, dx = \frac{3}{2} x^2 + 4x - \frac{1}{3} x^3 \bigg|_{-1}^{4} \]
\[ = \left( \frac{3}{2} \cdot 16 + 4 \cdot 4 - \frac{1}{3} \cdot 64 \right) - \left( \frac{3}{2} - 4 + \frac{1}{3} \right) \]
\[ = 44 - \frac{3}{2} - \frac{65}{3} = \frac{125}{6}. \]

1.2 Solids of rotation: Shells

The volume of a shell of radius \( R \) height \( h \) and thickness \( dx \) is
\[ dV = 2\pi R h \, dx. \]

Example 2 Find the area of the solid obtained by rotating about the y-axis the region bounded by \( f(x) = -x^2 + 5x - 6 \) and the x-axis.

Solution The graph of \( f(x) = -x^2 + 5x - 6 = (3 - x)(x - 2) \) is a downward opening parabola that intersects the x-axis at \( x = 2 \) and \( x = 3 \). At each value of \( x \) lies a test-rectangle of height \( f(x) \) and width \( dx \). This rectangle, when rotated about the y-axis, produces a shell of radius \( x \), height \( f(x) \), and width \( dx \). Thus each shell has volume
\[ dV = 2\pi f(x) x \, dx. \]

Summing up the infinitesimal volumes, we get
\[ V = \int_{2}^{3} dV \]
\[ = \int_{2}^{3} 2\pi (-x^2 + 5x - 6) x \, dx \]
\[ = 2\pi \int_{2}^{3} (-x^3 + 5x^2 - 6x) \, dx \]
\[ = 2\pi \left( -\frac{1}{4} x^4 + \frac{5}{3} x^3 - 3x^2 \right) \bigg|_{2}^{3} \]
\[ = \frac{\pi}{12}. \]