Lecture 25

Info about final:

* May 12 8:30pm - 11pm (Chike in Syllabus).
* Everyone will be on Zoom with webcam on (if no webcam, use phone).
* Will submit photos of work on gradescope.
* Allowed to use the textbook / hw / hw solns / notes / lecture notes. but no other sourse
* Focus will be on post spring break material, + some induction.

Recap

X countable $\iff |\mathbb{Z}^+| = |X|$

↑
set

$\iff \exists$ bijection

$f : \mathbb{Z}^+ \to X$

$\iff$ informally — Can list all the elements of X in a sequential list (possibly infinite)

E.g. $|\mathbb{Z}^+| = |E|$ ← positive even integers.

$1 \leftrightarrow 2$
$2 \leftrightarrow 4$
$3 \leftrightarrow 6$
$\vdots$

$|\mathbb{Z}^+| = |\mathbb{Z}|$

$\mathbb{Z}^+$: positive integers. $\mathbb{Z}$: all integers

$|\mathbb{Z}^+| = |\mathbb{Q}^+|$

$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{3}, \frac{3}{2}, \frac{3}{1}, \ldots$

There are infinite sets $|X|$ for which

$$|\mathbb{Z}^+| \neq |X|$$

<u>Theorem</u> Cantor (1891)

There are no surjections

$$\mathbb{Z}^+ \to P(\mathbb{Z}^+)$$

Proved at end of last lecture.

Another way of saying the result

There are different infinities.

$$|\mathbb{Z}^+| < |P(\mathbb{Z}^+)|$$

Another way of saying the result

Any attempt to list elements in $P(\mathbb{Z}^+)$ will not be exhaustive even if its an infinite list.

Contrast with the following:

Let X = The set ~~of~~ english texts.

$X = \{$ 'aaa', 'the', 'hello world', The textbook, lord of the rings'

--- , "asdfk~ ~~~ ~ ~",

Your biography": }.

X is countable.

Here is a list of all elements of X:

* List all sequences with a, b, c, ␣, etc of length 1
* List all sequences with a, b, c, ␣, etc of length 2
* List all sequences with a, b, c, ␣, etc of length 3

⋮

Any text will be seen in this list.

So X is countable, because we have exhibited a bijection $f: \mathbb{Z}^+ \to X$. (i.e. a listing of elements)

$f(1) = ''$
$f(2) = 'a'$
$f(3) = 'b'$
$f(4) = 'c'$

Contrast with $P(\mathbb{Z}^+)$.

Any attempt to list elements in $P(\mathbb{Z}^+)$ will not be exhaustive even if its an infinite list.

Note: in our list for X:

will see: a lot of math:

* "the set of even numbers"

* "the set containing one"

These are descriptions of things in $P(\mathbb{Z}^+)$.

But it will not contain descriptions of all elements.

$P(\mathbb{Z}^+)$.

<u>Consequence</u> Some subsets of $\mathbb{Z}^+$ cannot be described.

(via english text).
or any language

Consequence.

The set of things that can be described in english is countable.

Another observation:

$\{$Finite subsets of $\mathbb{Z}^+\}$ is countable.

$\{$subsets of $\mathbb{Z}^+\}$ is not uncountable.

$\{$infinite subsets of $\mathbb{Z}^+\}$ is not uncountable. ?

if

$\{\text{infinite subsets of } \mathbb{Z}^+\}$

were countable,

Finite subsets of $\mathbb{Z}^+$.

Then

$\{\text{infinite subsets of } \mathbb{Z}^+\} \cup P_*(\mathbb{Z})$

would be countable.

But this would mean

$P(\mathbb{Z})$ is countable.

We proved there is no surjection $\mathbb{Z}^+ \to P(\mathbb{Z}^+)$.

The proof actually shows: if X is a set, there is no surjection from $X \to P(X)$.

$n < 2^n$

<u>Consequence.</u>

$$|\mathbb{Z}^+| < |P(\mathbb{Z}^+)| < |P(P(\mathbb{Z}^+))| < |P(P(P(\mathbb{Z}^+)))| \cdots$$

Recall: I asked you to list $P(P(\{a,b\}))$.

16 elements.

There infinitely many infinities.

$$|\mathbb{Z}^+| < \;?\; < |\mathcal{P}(\mathbb{Z}^+)|$$

Is there another infinity in between?

This question was resolved in 1950s...

it's undecible

Theorem (Cohen)

It is impossible to prove or disprove this fact.

Other uncountable sets.

$\mathbb{R}$ = set of real numbers

$= \{1, 2, \pi, 2.718\ldots, \sqrt{2}, \ldots\}$.

Theorem: (Ch 14)

$[0,1]$ is uncountable.

(so $\mathbb{R}$ is uncountable).

Proof:

There is a surjection

$g: [0,1] \to P(\mathbb{Z}^+)$.

$g(0.1) = \{1\}$

even numbers

$g(0.0101010101\ldots) = E$

$g(0.11111111111\ldots) = \mathbb{Z}^+$

$g(x) = \{ i \in \mathbb{Z}^+ :$ i'th digit in binary expansion of x is $1 \}$.

Why is it a surjection?

$g(\underline{\quad\quad}) = A$

need to show for any A, we can put a number here.

Put $x = 0.(a_i)$

where $a_i = 1$ if $i \in A$.

Ex.

$$g(0.0100100001) = \{1, 4, 9\}.$$

So we have a surjection

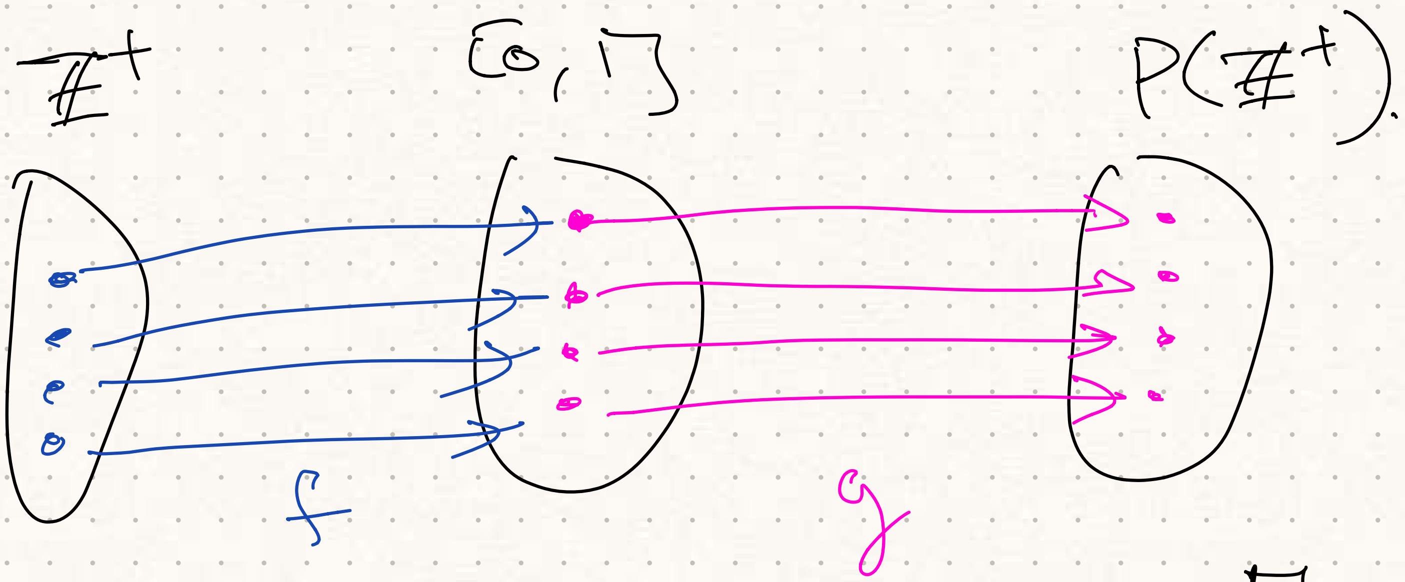
$$g : [0,1] \to P(\mathbb{Z}^+).$$

That means there is no surjection

$$f : \mathbb{Z}^+ \to [0,1].$$

Otherwise we could use it to get a surjection

$$\mathbb{Z}^+ \xrightarrow{f} [0,1] \xrightarrow[g]{} P(\mathbb{Z}^+)$$



<u>Consequence</u> Some real numbers cannot be described.

(via english text).

or any language