Lecture 24.

Recall

$|X| = |Y|$ means there is a bijection $f : X \to Y$

X 'countable' means

$$|\mathbb{Z}^+| = |X|$$

i.e. is there a bijection $f : \mathbb{Z}^+ \to X$.

$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$.

Last time: we looked at examples

$|\mathbb{Z}^+| = |E|$ $\qquad E = \{2, 4, 6, \ldots\}$

$$\begin{array}{ccccccc} (& 1 & 2 & 3 & 4 & \ldots & \mathbb{Z}^+ \\ & \downarrow & \downarrow & \downarrow & \downarrow & & \\ & 2 & 4 & 6 & 8 & & E \end{array}$$

$\times |\mathbb{Z}^+| = |\mathbb{Z}|$ $\mathbb{Z} = \{-1, 1, 2, -2, \ldots\}$

$0, 1, -1, 2, -2, 3, -3, \ldots$

$* \; |\mathbb{Z}^+| = |\mathbb{Q}^+|$ $\mathbb{Q}^+ = \{\frac{1}{3}, \frac{11}{13}, \frac{11}{500}, \ldots\}$

$0, \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{3}, \frac{3}{1}, \frac{3}{2},$

$\frac{1}{4}, \frac{3}{4}, \frac{4}{1}, \frac{4}{3}, \ldots$

$* \; |\mathbb{Z}^+| \overset{?}{=} |P(\mathbb{Z}^+)|$

$P(\mathbb{Z}^+) = \{\{1\}, \emptyset, \{1,2,3\}, E, \mathbb{Z}^+, \ldots\}.$

First attempt:

* $\emptyset$
* List all subsets of $\{1\}$,
* List all subsets of $\{1,2\}$
* List all subsets of $\{1,2,3\}$

$\vdots$

This fails, not going to have $\mathbb{Z}^+$, or E, or any infinite set.

Second attempt

* ϕ
* $\{1, 2, 3, \ldots\}$, $\mathbb{Z}^+$
* $\{1\}$
* $\{2, 3, 4, \ldots\}$
* $\{2\}$
* $\{1, 2\}$
* $\{1, 3, 4, 5, 6, \ldots\}$.
* $\{3, 4, 5, 6, \ldots\}$.

---

* List the empty set
* List, $\mathbb{Z}^+$
* List subsets of $\{1\}$
* List infinite subsets without subsets of $\{1\}$.

* $\{7, 71, 9\}$

* $\mathbb{Z}^+ - \{7, 71, 9\}$

Does this attempt work?

No, not going to hit

$\underline{\underline{\underline{E}}}$. (even numbers).

$$\mathbb{Z}^+ - O = E.$$

For every set X in this list, either X is finite, or $\mathbb{Z}^+ - X$ is finite.

So listing can never get E = even numbers.

(because E infinite, $\mathbb{Z}^+ - E$ also infinite).

In many cases, we have seen $|X| = |\mathbb{Z}^+|$

We get the feeling many sets are countable.

One way to make this precise: prove some general theorems.

Propisition 14.2.2

if A & B are countable,

$A \cup B$ is countable.

Proof:

Suppose A & B countable.

This means we can exhaustively list all the elements.

A: a_1 a_2 a_3 a_4 - - -

B: b_1 b_2 b_3 b_4 - - -

Then

$A \cup B$: a_1 b_1 a_2 b_2 a_3 b_3 ....

This is a similar idea to how we proved

$$|\mathbb{F}^+| = |\mathbb{F}|$$

$$\mathbb{F} = \mathbb{F}^- \cup \mathbb{F}^+$$

Prop 14.2.3

if A & B are countable

then $A \times B$ is countable.

Proof:

Suppose A & B is countable.

Put elements of $A \times B$ in grid.

$$\begin{array}{llll}
(a_1, b_4) & & & \\
(a_1, b_3) & & & \\
(a_1, b_2) & (a_2, b_2) & (a_3, b_2) & \cdots \\
(a_1, b_1) & (a_2, b_1) & (a_3, b_1) & \cdots
\end{array}$$

$A \times B$: (a_1, b_1) (a_1, b_2) (a_2, b_1) (a_1, b_3) $\cdots$

Similar to proof that

$$|\mathbb{Z}^+| = |\mathbb{Q}^+|.$$

<u>Theorem</u> (Cantor 1891).

$$|\mathbb{Z}^+| \neq |P(\mathbb{Z}^+)|.$$

We will actually prove:

<u>Theorem</u>

Let X be a set.

There is no surjection

$$f : X \to P(X).$$

This theorem is why we don't say $|\mathbb{Z}^+| = \infty$

$$|P(\mathbb{Z}^+)| = \infty$$

Because then we would have

$$|\#^+| = |P(\#^+)|$$

(but not bijection $\#^+ \to P(\#^+)$).

Warmup (Preparation for proof).

<u>Quiz</u>

Let A and B sets
x be an element.

if $x \in A$
$x \notin B$.

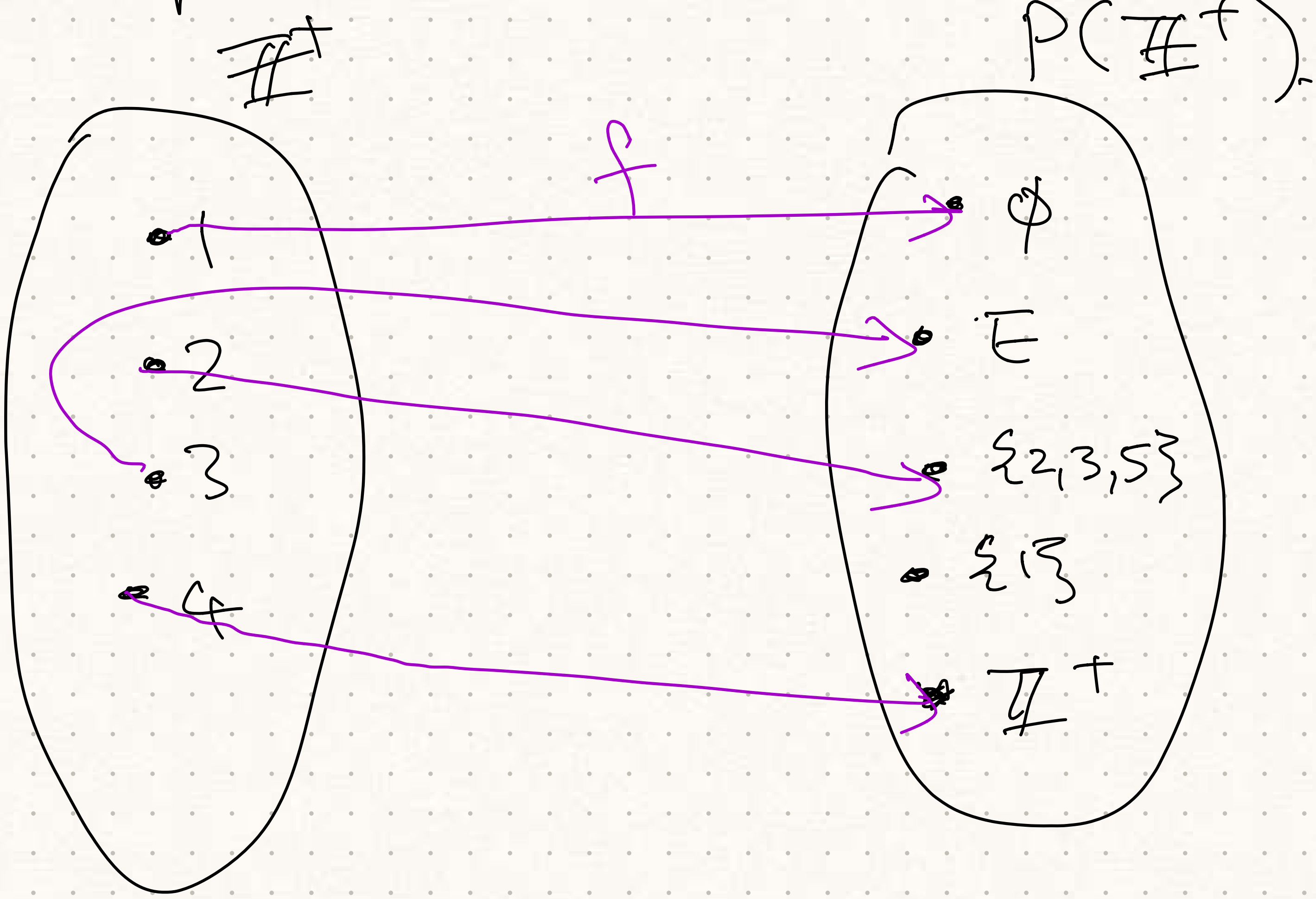
$A = B$ T/F?

$A = \{x, \quad\}$

$B = \{ \quad \}$

x is in A
but not in B, $A \neq B$.

Warmup 2:



Let $A = \{x \in \mathbb{Z}^+ : x \notin f(x)\}$.

- $1 \in A$ T
- $2 \in A$ F
- $3 \notin A$ F $(3 \in A)$.
- $4 \notin A$ T.

* $4 \notin A$ T/F?

$4 \notin f(4)$

$4 \notin \mathbb{Z}^+$

F

(Because

$4 \in A$

$4 \in f(4)$

$4 \in \mathbb{Z}^+$

T).

Warmup 7

if $x \in A$

$x \notin B$

then $A \neq B$

Theorem

There is no surjection

$$f: \mathbb{Z}^+ \to P(\mathbb{Z}^+).$$

Proof

Let $f: \mathbb{Z}^+ \to P(\mathbb{Z}^+)$

be a function.

Let

$$A = \{x \in \mathbb{Z}^+ : x \notin f(x)\}.$$

if $x \in A$ then $x \notin f(x)$ | $\to f(x) \neq A$

if $x \notin A$ then $x \in f(x)$ | $\to f(x) \neq A$.

So, in all cases

$f(x) \neq A$.

So f is not surjective. $\blacksquare$