Lecture 23

HW Due Monday

Last time: if X and Y are sets

Defined

* "$|X| = |Y|$" means $\exists f: X \to Y$ bijection

* "$|X| \leq |Y|$" means $\exists f: X \to Y$ injective

What about

multiple choices.

* "$|X| \geq |Y|$" means

  $\exists f: X \to Y$ surjection

  $\exists f: Y \to X$ injection

Take this one.

Warning: $|X|$ has no meaning

e.g. $|\mathbb{Z}^+|$ has no meaning

__Example__ $\quad \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$

$E = \{2, 4, 6, \ldots\}$.

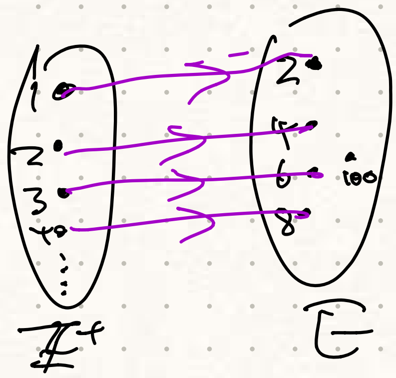
* $|\mathbb{Z}^+| \overset{?}{=} |E|$

i.e. is there a bijection $f : \mathbb{Z}^+ \to E$?

i.e. is there a $f : \mathbb{Z}^+ \to E$? which is surjective and injective.

Yes.

* $f(x) = 2x$

* 

__injective:__

if $x \neq y$,

$2x \neq 2y$ ✓

__surjective:__

given $y \in E$, ✓

$f\left(\frac{y}{2}\right) = y$.

We've just shown that

$$|\mathbb{Z}^+| = |E|.$$

Quiz:

1) $E \subset \mathbb{Z}^+$ T / F   80%

True, $x \in E \Rightarrow x \in \mathbb{Z}^+$

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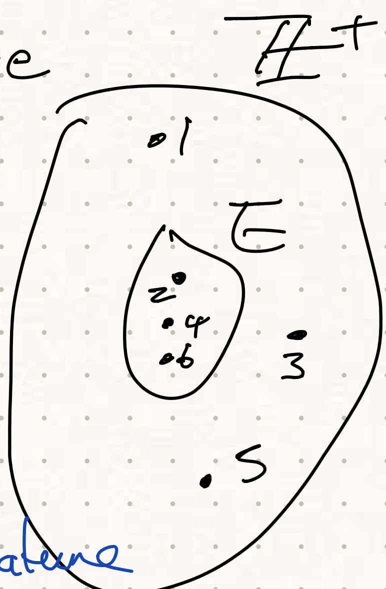
2) $E = \mathbb{Z}^+$ T / F   66%.

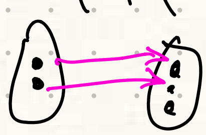
False, $1 \in \mathbb{Z}^+$, but $1 \notin E$.

So we have
$E \subset \mathbb{Z}^+$
$E \neq \mathbb{Z}^+$
but
$|E| = |\mathbb{Z}^+|$



Contrast this to the situation for finite sets:

$A \subset B$
$A \neq B$
Then
$|A| < |B|$.



This is a surprising feature of cardinalities for infinite sets.

$*|\mathbb{Z}^+| \overset{?}{=} |\mathbb{Z}|$ $\mathbb{Z} = \{\dots,-2,-1,0,1,2,\dots\}$.

Is there a bijection

$$f: \mathbb{Z}^+ \to \mathbb{Z} \; ?$$

Yes

$$f(x) = \begin{cases} \frac{x-1}{2} & x \text{ odd} \\ -\frac{x}{2} & x \text{ even.} \end{cases}$$

E.g.

$f(1) = 0$

$f(2) = -1$

$f(3) = 1$

$f(4) = -2$

$f(5) = 2$

$f(6) = -3$

$f(7) = 3$

$\vdots \quad -4$

$\quad\;\; 4$

$\quad -5$

$\quad\;\; 5$

$\quad\;\; \vdots$

injective?

Plausible ✓

surjective

Plausible ✓.

So (plausibly)

bijective. ✓.

How to formally prove that f is a bijection?

Easiest way: show that f has an inverse.

Here, $g: \mathbb{Z} \to \mathbb{Z}^+$,

$$g(x) = \begin{cases} -2x & x < 0 \\ 2x+1 & x \geq 0. \end{cases}$$

E.g. $g(3) = 2 \cdot 3 + 1 = 7$.

Then verify

$$g(f(x)) = x \qquad f(g(x)) = x.$$

Note on notation:

To describe a function

$$f: \mathbb{Z}^+ \longrightarrow X,$$

we can just "list" the function values.

E.g.

The first example $f: \mathbb{Z}^+ \to E$ can be written

$$\underset{f(1)}{2}, \underset{f(2)}{4}, \underset{f(3)}{6}, \underset{f(4)}{8}, \underset{f(5)}{10}, \ldots$$

The second example $f: \mathbb{Z}^+ \to \mathbb{Z}$.

$$0, -1, 1, -2, 2, -3, 3, \ldots$$

With this notation, we have the following dictionary:

function $\mathbb{Z}^+ \to X$ $\Longleftrightarrow$ infinite sequence of elements in X

surjective $\Longleftrightarrow$ every $y \in X$ can be found in the list

injective $\Longleftrightarrow$ no duplicates in list.

$\mathbb{Q}^+ = \{0, \frac{1}{2}, \frac{1}{3}, \frac{3}{7}, \frac{11}{29}, \ldots\}$.

ratios of positive integers (and 0).

So far:

$|\mathbb{Z}| = |\mathbb{Z}^+|$

$|\mathbb{Z}| = |E|$.

- $|\mathbb{Z}^+| \stackrel{?}{=} |\mathbb{Q}^+|$

Let's attempt find bijection:

$f : \mathbb{Z}^+ \longrightarrow \mathbb{Q}^+$

$f(x) = \frac{1}{x}$

a) injective ✓

b) surjective ✗

c) bijective ✗

- $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \ldots$

a) it is injective. no duplicates.

b) not surjective,

~~$\frac{3}{7}$,~~ 2 is not in list

$0 \in \mathrm{Im}\, f$ (i.e. is 0 in list).

No, not in list.

(even though "limit" is 0).

To say that something is in list, must be able to (theoretically) tell <u>what position</u> it occurs.

E.g. $\frac{1}{100}$ is at position 100.

E.g. 0 is at position ___?

<u>Another attempt:</u>

• $0, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \ldots$

Not surjective, missing 2.

• $0, \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{1}, \frac{2}{3}, \frac{3}{2}, \frac{1}{4}, \frac{4}{1}, \frac{3}{4}, \frac{4}{3}, \ldots$

Will I get $\frac{572}{999}$?

Will I get $\frac{p}{q}$?

Yes, wait until listing numbers with denominator q:

$$\frac{1}{q}, \frac{q}{1}, \frac{2}{q}, \frac{q}{2}, \ldots \frac{p}{q}, \frac{q}{p}$$

So it's surjective and injective, so we have a bijection.

So $|\mathbb{Z}^+| = |\mathbb{Q}^+|$

$|\mathbb{Z}^+| = |\mathbb{Z}|$
$|\mathbb{Z}^+| = |\mathbb{Z}|$
$|\mathbb{Z}| = |\mathbb{Q}^+|$

Example

$$|\mathbb{Z}^+| \stackrel{?}{=} |P(\mathbb{Z}^+)|$$

Recall $P(\mathbb{Z}^+) =$ set of subsets of $\mathbb{Z}^+$

$$= \{\phi, \{1\}, \{2\}, \mathbb{Z}^+\}.$$

Is there a bijection? $f : \mathbb{Z}^+ \to P(\mathbb{Z}^+)$.

Attempt:

$\phi, \{1\}, \{2\}, \{1,2\}, \{3\}, \{1,3\}, \{2,3\},$
$\{1,2,3\}, \{4\}, \ldots$

Or:

* List all subsets of ϕ
* List subsets of $\{1\}$ (that you haven't seen yet)

Or:

0) List all subsets of $\emptyset$

1) List subsets of $\{1\}$ (that you haven't seen yet)

2) List subsets of $\{1,2\}$ (that you haven't seen yet)

3) List subsets of $\{1,2,3\}$, that you haven't seen yet)

$\vdots$

8) List subsets of $\{1, \dots, 8\}$ that you haven't seen yet)

Will you see $\{5, 6, 8\}$ in the list?

Yes, in step 8.

Will you see $\{2, 4, 6, 8, \dots, 10000\}$?

Yes in step 10,000.

Is f surjective?

$$f : \mathbb{Z}^+ \to P(\mathbb{Z}^+)$$

Ans: Yes / No

27%.

Which element of $P(\mathbb{Z}^+)$ is not in the list?

(not in Im f).

$\{1, 2, \dots\}$.

$\mathbb{Z}^+$ & Im f

Because everything in our list is finite.

Im f is missing all infinite sets.

$P(\mathbb{Z}^+) =$

- $\emptyset$
- $\{1\}$
- $E = \{2, 4, 6, 8, \ldots\}$
- $\mathbb{Z}^+ = \{1, 2, \ldots\}$

<u>2 things:</u>

1) Fix our attempt

2) Go back to $f : \mathbb{Z}^+ \to \mathbb{Q}^+$

ex.
describe like, step by step process.