Lecture 22

HW 8 out, due next Monday 3pm.

First 3 problems should be doable from last week.

Poll: (For future, 2021)

a) Pay \$3 per class to have assignments / exams thru gradescope

b) Do not \$3, hw/exams done normally (paper).

Example

Prove

$$\binom{n+1}{2} = 1 + 2 + \cdots + n$$

We will exhibit a bijection

$f: P_2(\mathbb{N}_{n+1}) \to \underline{\{1\}\times\mathbb{N}_1} \cup \underline{\{2\}\times\mathbb{N}_2} \cup \cdots \cup \{n\}\times\mathbb{N}_n$

$\{ (1,1) \}$ $\qquad \{1\}\times\mathbb{N}_1$

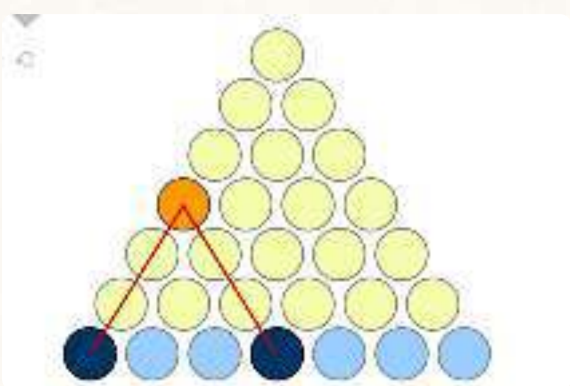
$\{ (1,1), (1,2) \}$ $\qquad \{2\}\times\mathbb{N}_2$

$\{(3,1), (3,2), (3,3)\}$ $\qquad \{3\}\times\mathbb{N}_3$

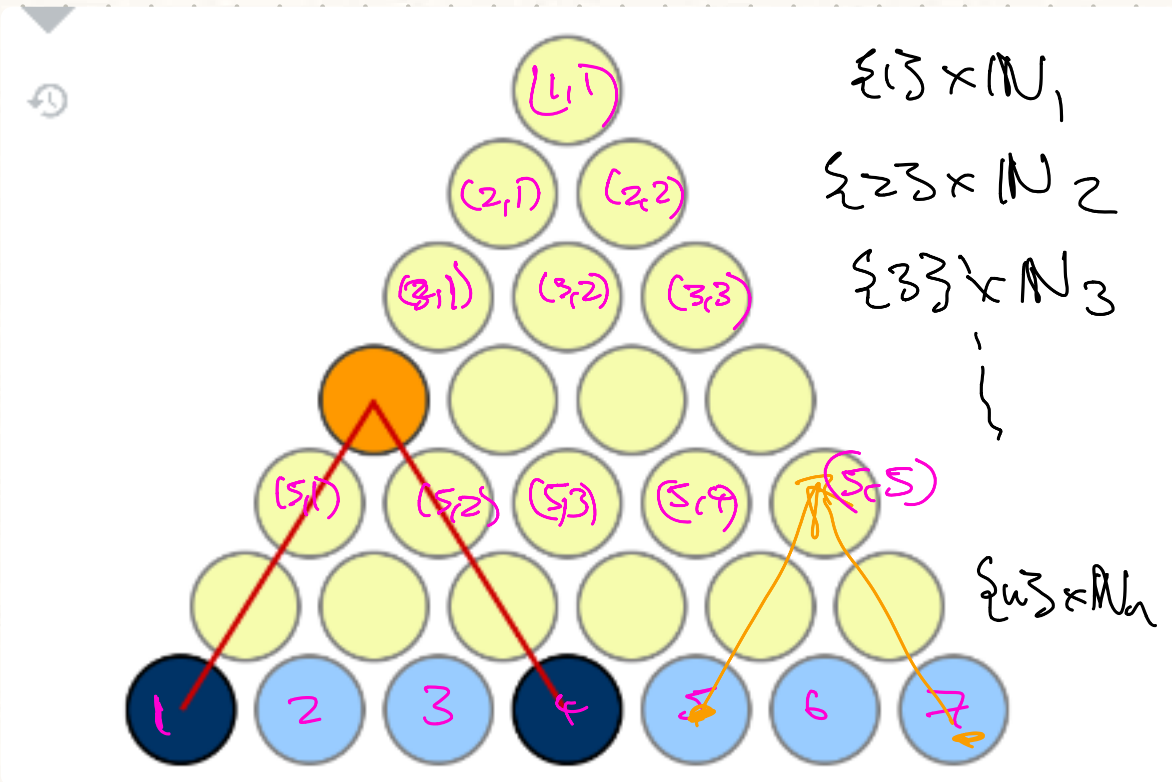
$\vdots$

$\{(n,1), (n,2), \ldots, (n,n)\}.$ $\qquad \{n\}\times_n\mathbb{N}$

$P_2(\mathbb{N}_{n+1})$



The animation shows the bijection.



$f: \mathcal{P}_2(\mathbb{N}_{n+1}) \to$ ______________________

f: "Shooting rays up".

f^{-1}: "shooting rays down".

E.g. $f(\{7,5\}) =$

a) $(6,1)$
b) $(6,2)$
c) $(3,2)$
d) $(4,5)$
e) $(6,6)$

$f(\{4,5\}) =$

a) $(6,2)$

b) $(2,3)$

c) $(6,4)$ ← 91%.

d) $(1,5)$

e) $(2,2)$

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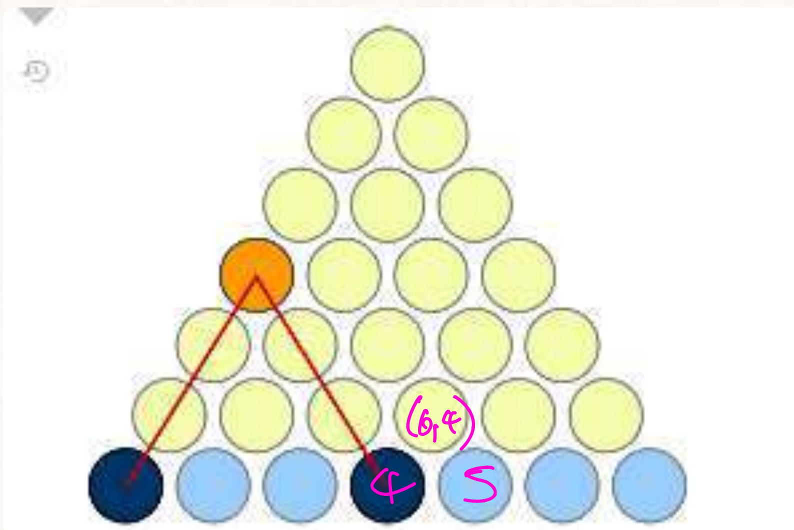
c) is correct.



More formally:

① Write down a formal description of f.

$$f(\{a,b\}) = (n+1-|a-b|, \min(a,b))$$

E.g. $f(\{7,5\}) = (6+1-2, 5) \neq (5,5).$

Inverse:

$$g((p,q)) = \{ q \quad , \quad \underline{\qquad\qquad} \}$$

exercise.

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(2) Verifying

$f \circ g = \mathrm{Id}$

$g \circ f = \mathrm{Id}.$

So

$$\binom{n+1}{2} = 1 + 2 \cdots\cdots + n$$

Bijective method

To prove an equality

$$a = b$$

1) Interpret LHS as the cardinality of some set A.

$$|A| = a$$

2) Interpret RHS as the cardinality of some set B.

$$|B| = b.$$

3) Find a bijection

$$f : A \to B.$$

Then conclude $|A| = |B|$. so $a = b$.

$$1 + 2 + \cdots + n = \binom{n+1}{2}$$

$$\frac{n(n+1)}{2} = \frac{n\cancel{(}n+1)}{2}$$

How to use ph. principle

Want to prove S.

1) Choose A (set)

2) Choose B (set)

$$|B| < |A|$$

3) Choose $f : A \to B$

(by PHP, f is not injective).

4) Explain why knowing f is not injective proves S.

Example (P4 HW7)

Proof

1) $A = X$ (46 elems)

2) $B = \{ \{1, 90\}, \{2, 89\},$

$\vdots$

$\{45, 46\} \}.$

$|B| = 45$, so $|A| > |B|$.

3) $f : A \to B$

$f(x) =$ the set containing x

e.g.

$f(3) = \{3, 88\}.$

$f(51) = \{51, 40\}$

$f(54) = \{54, 37\}$

4) f is not injective.

$\exists\, x, y \in X$, $x \neq y$, $f(x) = f(y)$

$\Rightarrow$ the set containing x

$=$

the set containing y.

$\Rightarrow$ x, y are consecutive

because we chose B properly.

Meta-Lesson

* Try to extract structure
* re-organize things.

Excercise: go back to all the pigeonhole examples (hair on head, points on sphere, point in square) .. etc.. and make it fit into framework.

Excercise: do the same thing for bijective proofs last lecture and today.

Counting infinite sets Ch 14.

Let

$\mathbb{Z}$ = Integers

$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$

$\mathbb{Q}$ = Rationals $= \{\frac{1}{2}, \frac{1}{3}, \ldots\}$

$E = \{2, 4, 6, 8, 0, -2, -4, \ldots\}$

$E^+ = \{2, 4, 6, 8, \ldots\}$.

Quiz

$$|\mathbb{Z}| = |E| \ ?$$

Correct answer: "I don't know"

Trick question. '$|\mathbb{Z}|$ undefined'

Recall: $|A| = n$ if there is a bijection $f : \mathbb{N}_n \to A$.

Since there are are no bijections

$$f : \{1, \ldots, n\} \to \mathbb{Z},$$

so $|\mathbb{Z}|$ is undefined.

<u>Definition</u>

A is finite if $|A|$ exists. (i.e. if

$$\exists n, \exists f : \mathbb{N}_n \to A \text{ bijection.}$$

<u>Näive solution:</u>

If A not finite, just declare

$$|A| = \infty.$$

Analogy:

if $|A| \geq 100$

just declare

$|A| = \infty$ ← Missing out on interesting behavior

Definition

if X, Y are sets.

* $|X| = |Y|$ means

there is a bijection
$f: X \to Y$

* $|X| \leq |Y|$ means
there is a injection.
$f: X \to Y$