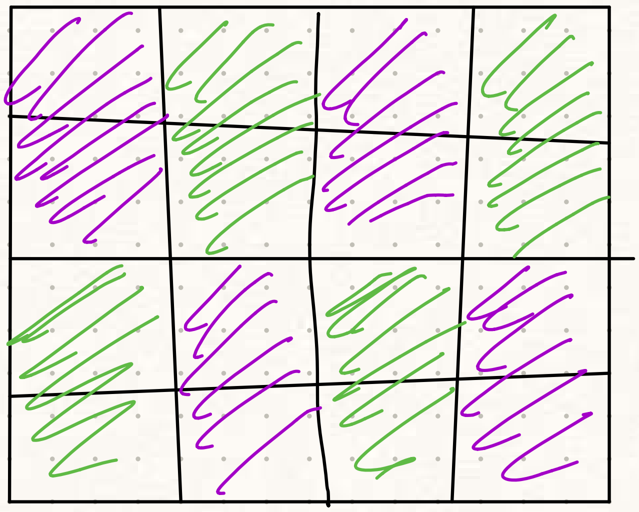
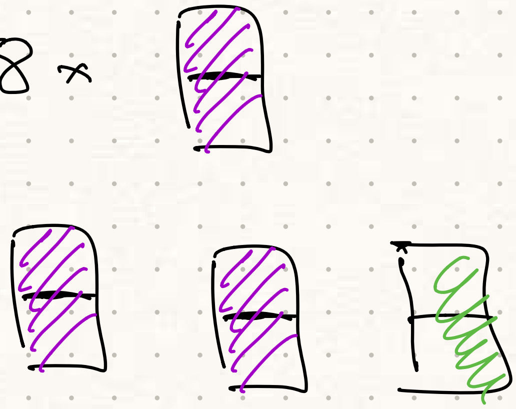
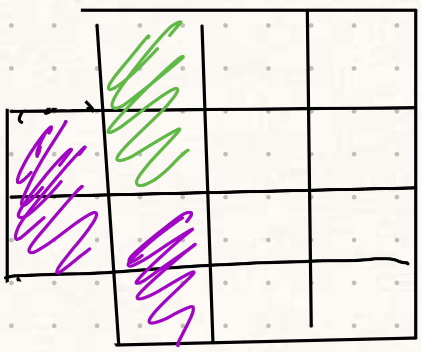
Lecture 21



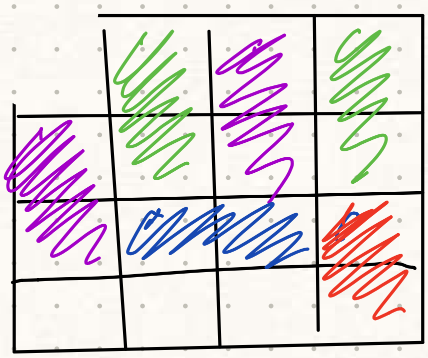
8 x





1) How many dominoes?

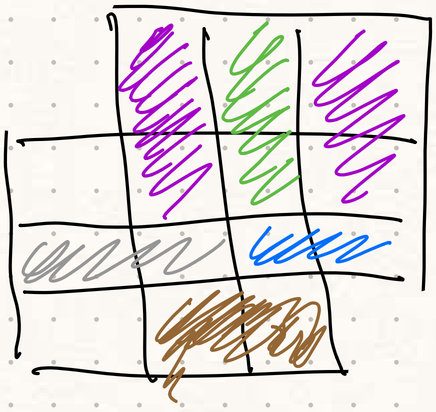
Can tile. ✓



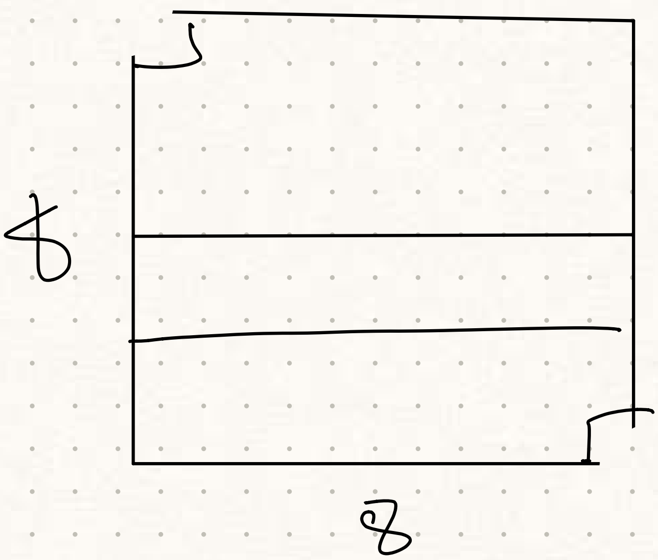
Can I tile?

T / F

<u>No</u>



Can I tile? with 7 dominoes



with 31 dominoes.

Recall that

* $P_r(\mathbb{N}_n)$ = subsets of $\{1, \dots, n\}$ with r elements

* $\binom{n}{r} = |P_r(\mathbb{N}_n)|$

Quiz:

$\binom{4}{2} =$

a) 2
b) 3
c) 4
d) 5
e) 6

$\binom{4}{2} = |P_2(\mathbb{N}_4)| =$ number of subsets of $\{1,2,3,4\}$ with 2 elements.

number of elents in

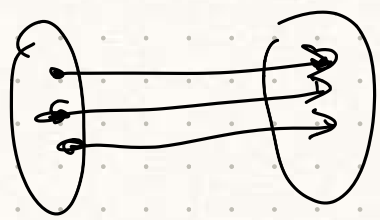
$= \{ \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\} \}$

$= 6.$

Last time:

* $\binom{n}{r} = \binom{n}{n-r}$ by exhibiting a bijection

$f: P_r(\mathbb{N}_n) \to P_{n-r}(\mathbb{N}_n)$



Theorem:

For $0 \le r \le n$

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}.$$

What does this have to do with pascal's triangle

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

$$\begin{tikzcd}
\binom{n-1}{r-1} \arrow[dr] & & \binom{n-1}{r} \arrow[dl] \\
 & \binom{n}{r} &
\end{tikzcd}$$

Proof

We will find a bijection

$$P_r(\underbrace{\mathbb{N}_n}_{\{1,\dots,n\}}) \to \left(P_r(\underbrace{\mathbb{N}_{n-1}}_{\{1,\dots,n-1\}}) \cup P_{r-1}(\underbrace{\mathbb{N}_{n-1}}_{\{1,\dots,n-1\}})\right)$$

Informally:

if $A \in P_r(\mathbb{N}_n)$, this means that $A \subset \{1,\dots,n\}$, with r elements.

$(1,0,1,0)$

$\updownarrow$

$\{1,3\}$.

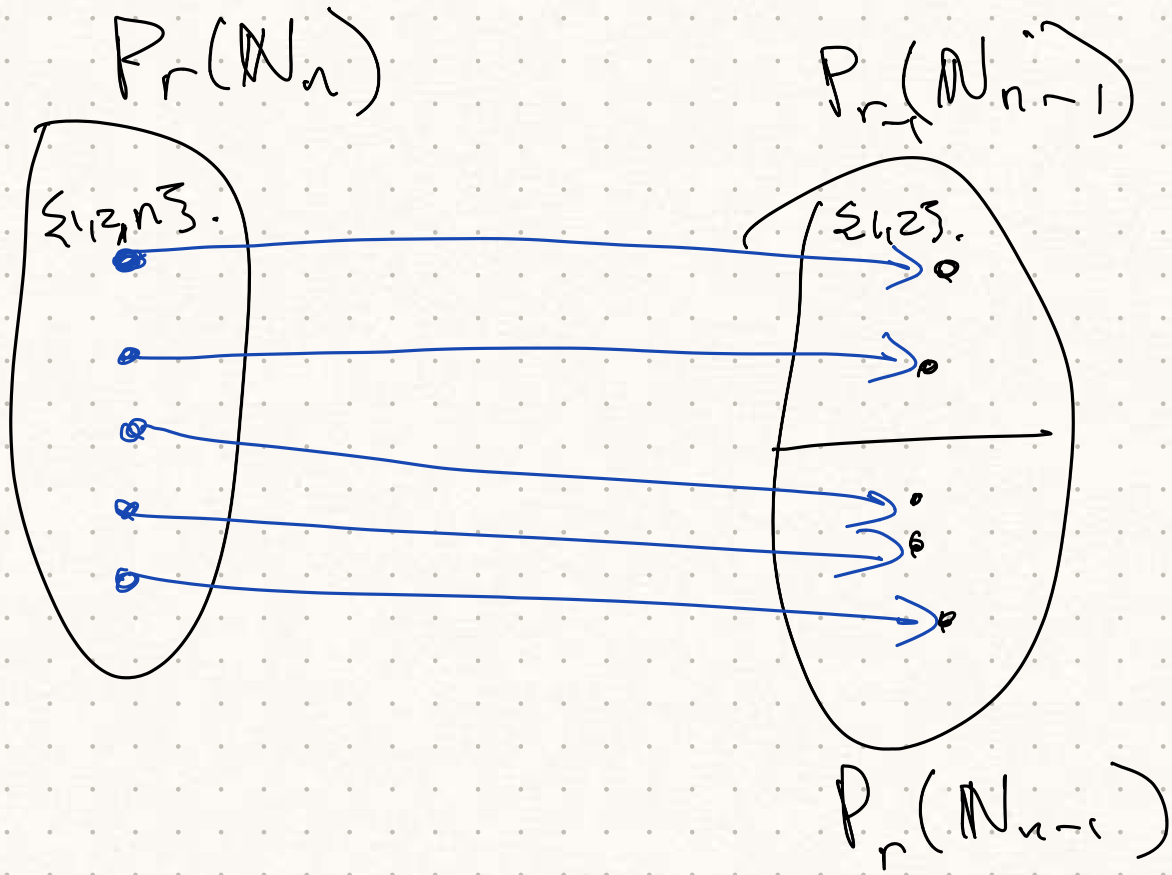
2 cases:

if $n \in A$, then A can be interpreted as an element of

$$P_{r-1}(\{1,\dots,n-1\}).$$

if $n \notin A$, then

$$A \in P_r(\{1,\dots,n-1\})$$



Formal Proof:

Define

$$f: P_r(\mathbb{N}_n) \to P_r(\mathbb{N}_{n-1}) \cup P_{r-1}(\mathbb{N}_{n-1})$$

$$f(A) = \begin{cases} A - \{n\} & \text{if } n \in A \\ A & \text{if } n \notin A. \end{cases}$$

To show it's a bijection,
define

$$g: P_r(\mathbb{N}_{n-1}) \cup P_{r-1}(\mathbb{N}_{n-1}) \longrightarrow P_r(\mathbb{N}_n)$$

$$g(B) = \begin{cases} B \cup \{n\}. & B \in P_{r-1}(\mathbb{N}_{n-1}) \\ B. & B \in P_r(\mathbb{N}_n). \end{cases}$$

Then $g(f(A)) = A$

$f(g(B)) = B$ (excercise)

So f has an inverse.

So f is a bijection.

So

$$|P_r(\mathbb{N}_n)| = |P_r(\mathbb{N}_{n-1}) \cup P_{r-1}(\mathbb{N}_{n-1})|$$

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}.$$

($|X| + |Y| = |X \cup Y|$ only if disjoint)

$$\begin{tikzcd}
\text{set } 1 \arrow[r, leftrightarrow, "\text{bijection}"] \arrow[d] & \text{set } 2 \arrow[d] \\
\text{expression } 1 \arrow[r, leftrightarrow, "\text{equality of numbers}"] & \text{expression } 2.
\end{tikzcd}$$

Theorem

For $n \geq 0$,

$P(n)$

$$\boxed{\text{For } 0 \leq r \leq n, \quad \binom{n}{r} = \frac{n!}{r!(n-r)!}}$$

where $n! = n(n-1) \cdots 1$.

e.g. $4! = 4 \cdot 3 \cdot 2 \cdot 1$

Proof by induction.

inductive step: $P(n) \Rightarrow P(n+1)$

by thm.

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!}$$

$$\binom{n+1}{r} \overset{\text{by thm.}}{=} \binom{n}{r} + \binom{n}{r-1} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!}$$

$$= \frac{(n+1)!}{r!\,(n+1-r)!}$$

common denominator/ algebra

Theorem

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

Not obvious from $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

E.g. $1+3+3+1 = 8 = 2^3$.

Proof:

$$P_0(\mathbb{N}_n) \cup P_1(\mathbb{N}_n) \cup \cdots \cup P_n(\mathbb{N}_n) = P(\mathbb{N}_n)$$

So

$$|P_0(\mathbb{N}_n) \cup P_1(\mathbb{N}_n) \cup \cdots \cup P_n(\mathbb{N}_n)| = |P(\mathbb{N}_n)|$$

$$|P_0(\mathbb{N}_n) \cup P_1(\mathbb{N}_n) \cup \cdots \cup P_n(\mathbb{N}_n)| = |P(\mathbb{N}_n)|$$

$$|P_0(\mathbb{N}_n)| + |P_1(\mathbb{N}_n)| + \cdots + |P_n(\mathbb{N}_n)| = |P(\mathbb{N}_n)|$$

So by defn,

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

Another example where
"interpreting numbers as a count"

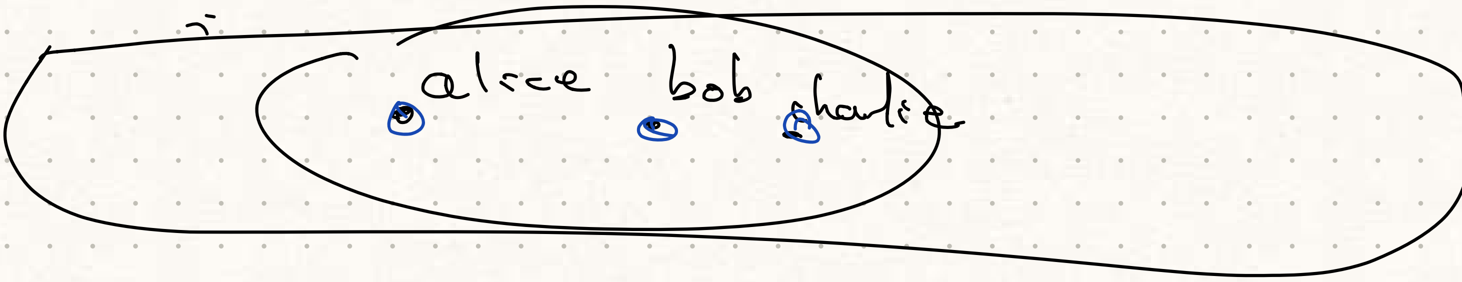
$$0\binom{n}{0} + 1\binom{n}{1} + 2\binom{n}{2} + \cdots n\binom{n}{n} = n \cdot 2^{n-1}$$

Interpretation:

$\binom{n}{r}$ = number of ways to choose r people from n candidates to send to Mars.

$$r\binom{n}{r} =$$

$r\binom{n}{r}$ = number of ways to
choose r people from
n candidates
to send to Mars
<u>and</u>
to pick a leader.



$0\binom{n}{0}+1\binom{n}{1}+2\binom{n}{2}+ \cdots\ n\binom{n}{n}$

= number of ways to
choose people from
n candidates
to send to Mars
<u>and</u>
to pick a leader.

=

equivalently:

pick a leader x (n choices)

pick rest of the team

(pick subset of $\{1,2,\dots,n\}-\{x\}$

(2^{n-1} choices)

"number of ways to choose people from n candidates to send to Mars" $= n2^{n-1}$

Formally, there is a bijection

$$f : N_1 \times P_1(N_n) \cup N_2 P_2(N_n) \cup \dots \cup N_n(N_n) \longrightarrow N_n \times P(N_{n-1}).$$