Lecture 20

We know:

* What is counting
* Why to count
* Shortcuts for counting

$$|X \cup Y| = |X| + |Y| - |X \cap Y|$$

$$|X \times Y| = |X| \cdot |Y|$$

Plan for rest of semester:

* Finish talking about finite.

$$f : \underset{\{1,\dots,n\}}{\mathbb{N}_n} \to \mathbb{R}$$

can't be bijection, it will not be surjective.

highlight: pigeonhole principle.

if $|A| > |B|$

then $f : A \to B$ is <u>never</u> injective

highlight: ways to explain identities like

$$\boxed{1 + 2 + \dots + n = \frac{n(n+1)}{2}}$$

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

"Bijective proofs"

Talking about "counting" infinite sets.

highlight: Cantor's Theorem (1891)

Ch 14.

Last time:

* $|X \cup Y| = |X| + |Y| - |X \cap Y|$
* $|X \times Y| = |X| \cdot |Y|$

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* $|X^n| = |X|^n$

X^n = n-tuples with entries in X.

Example: $\{0,1\}^3 = \{0,1\} \times \{0,1\} \times \{0,1\}$

$$= \{ (0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1) \}$$

Recall
$\ast$ $P(X)$ = set of subsets of X.

E.g.
$P(\emptyset) = \{\emptyset\}$ $\swarrow$ 1

$P(\{a\}) = \{\emptyset, \{a\}\}$ $\swarrow$ 2

$P(\{a,b\}) = \{\{a\}, \{b\}, \emptyset, \{a,b\}\}$ $\swarrow$ 4

$P(\{a,b,c\}) = \{\emptyset, \{c\}, \{a\}, \{a,c\}, \ldots\}$ $\swarrow$ 8

$P(\{a,b,c,d\}) = \{ \qquad , \qquad , \qquad 16 = 2^4 \}$

Conjecture
$$|P(Y)| = 2^n, \text{ where } n = |Y|.$$

That was informal explanation.

$$f(A) = \overbrace{(\qquad\qquad\qquad)}^{n \text{ entries}}$$

$$i\text{th coordinate} = \begin{cases} 0 & \text{if } i \notin A \\ 1 & \text{if } i \in A \end{cases}$$

Example

$n = 4$

$A = \{2, 3\}.$

$$f(A) = \overbrace{(\qquad\qquad\qquad)}^{n \text{ entries}}$$

$$i\text{th coordinate} = \begin{cases} 0 & \text{if } i \notin A \\ 1 & \text{if } i \in A \end{cases}$$

$$= \overbrace{(\qquad\qquad\qquad)}^{4 \text{ entries}}$$

$$i\text{th coordinate} = \begin{cases} 0 & \text{if } i \notin \{2,3\} \\ 1 & \text{if } i \in \{2,3\} \end{cases}$$

$$= (0, 1, 1, 0)$$

Example

$n = 4$

$A = \{1, 2\}$

$f(A) =$

$\subseteq \{1, 2, 3, 4\}.$

a) $(1, 1, 1, 1)$

b) $(1, 1, 0, 0) \leftarrow \{1, 2\}$

c) $(0, 0, 1, 1) \leftarrow \{3, 4\}.$

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is $f : P(\cancel{X}) \to \{0, 1\}^n$.

is it injective?

if $A \neq B$, $f(A) \neq f(B)$?

Yes proof: Let i be an element of $A - B$ (or $B - A$).

Then the ith coordinate of $f(A)$ is 1, but ith coord of $f(B)$ is 0.

So $f(A) \neq f(B)$ as desired.

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f is surjective:

given a vector

$v \in \{0,1\}^n$ ...

want A such that

$f(A) = v$.

In the example, $f(\{1,3,4\}) = (1,0,1,1)$.

Let

$A = \{ i \in Y :$ i'th coordinate of v is $1 \}$.

So $f : P(Y) \to \{0,1\}^n$ is bijective

So

$$|P(Y)| = |\{0,1\}^n| = 2^n.$$

$\blacksquare$

Defn

* For $0 \le r \le n$ integer,

$$P_r(X) = \{A \subseteq X : |A| = r\}.$$

* $\binom{n}{r} = |P_r(\mathbb{N}_n)| \overset{\text{Theorem}}{=} \underbrace{\qquad\qquad}_{\text{we don't know this yet.}}$

Example

$|P_2(\mathbb{N}_3)| =$

a) 0

b) 1

c) 2 ← 50%

d) 3 ✓ ← 50%

e) 4

$$P_2(\mathbb{N}_3) = \{A \subseteq \{1,2,3\} : |A| = 2\}.$$

$$= \{\{1,2\}, \{3,2\}, \{1,3\}\}.$$

Let $X = \{1,2,3\}$

$$P_0(X) = \{\phi\}.$$

$$P_1(X) = \{\{1\}, \{2\}, \{3\}\}.$$

$$P_2(X) = \{\{1,2\}, \{3,2\}, \{1,3\}\}$$

$$P_3(X) = \{\{1,2,3\}\}.$$

Prop 12.2.6

1) $\binom{n}{0} = |P_0(\mathbb{N}_n)| = 1.$

Proof: $P_0(\mathbb{N}_n) = \{\phi\}.$

2) $\binom{n}{1} = n$

Proof: There is a bijection $f: P_1(\mathbb{N}_n) \to \mathbb{N}_n$.

3)

$$\binom{n}{r} = \binom{n}{n-r}$$

Proof of 3):

We will construct a bijection

$$f: P_r(\mathbb{N}_n) \longrightarrow P_{n-r}(\mathbb{N}_n)$$

Example

$$P_1(X) = \{\{1\}, \{2\}, \{3\}\},$$

$$P_2(X) = \{\{1,2\}, \{3,2\}, \{1,3\}\}$$

Let $f(A) = \mathbb{N}_n - A$.

Why is it a bijection?

It has an inverse

$$g: P_{n-r}(\mathbb{N}_n) \to P_r(\mathbb{N}_n)$$

$$g(B) = \mathbb{N}_n - B.$$

We have

$$f(g(B)) = \mathbb{N}_n - (\mathbb{N}_n - B)$$

$$= B$$

Exercise - $X = Y$ means $x \in X \Leftrightarrow x \in Y$.

Also

$$g(f(A)) = \mathbb{N}_n - (\mathbb{N}_n - A)$$

$$= A.$$

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

Theorem 12.2.8.

For $1 \leq r \leq n$ integer

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

Proof:

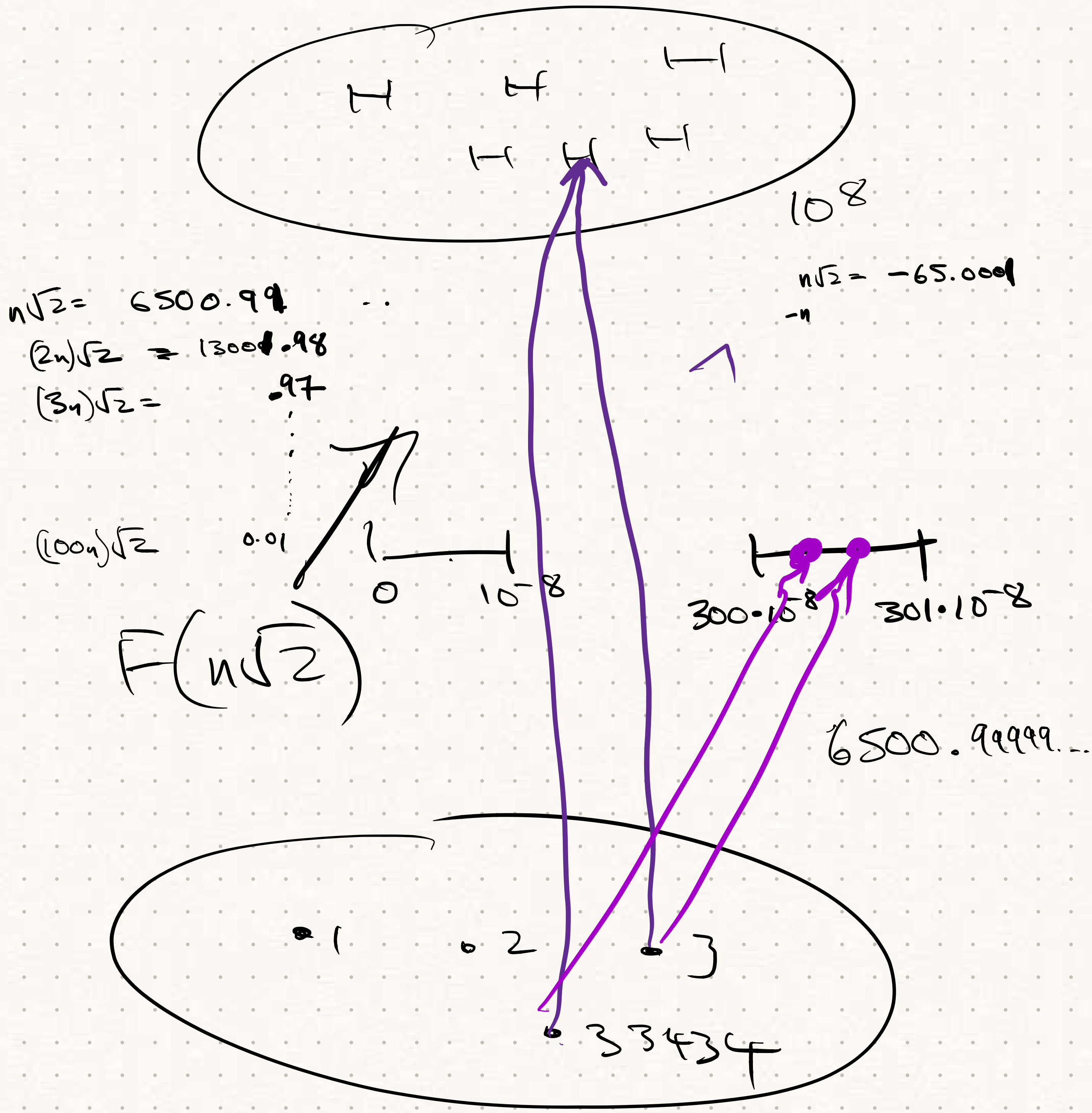
We will construction a bijection

$$f : P_r(\mathbb{N}_n) \to P_{r-1}(\mathbb{N}_{n-1}) \cup P_r(\mathbb{N}_{n-1})$$

(because then

$$\underbrace{|P_r(\mathbb{N}_n)|}_{\binom{n}{r}} = |P_{r-1}(\mathbb{N}_{n-1}) \cup P_r(\mathbb{N}_{n-1})|$$

$$= \underbrace{|P_{r-1}(\mathbb{N}_{n-1})|}_{\binom{n-1}{r-1}} + \underbrace{|P_r(\mathbb{N}_{n-1})|}_{\binom{n-1}{r}}$$



$$\boxed{|F(m\sqrt{2}) - F(m'\sqrt{2})| < 10^{-8}}$$

so

$$m\sqrt{2} = 305.5100001$$
$$m'\sqrt{2} = 7124.5100000$$

$$\underbrace{(m-m')}_{n}\sqrt{2} = -6500.00001$$