Homework 7 due date to Next monday 20 April 3pm

Hints for HW.

Name: Replace this with your name.

Problem 1 (10 points)

The following is a proof that if f is a bijection, then f has an inverse. The author has assumed that the reader can justify each step. Your task is to justify each step.

Proof:

1. Suppose $f : X \to Y$ is a bijection.
2. Define $g : Y \to X$ as follows. For $y \in Y$, let
$$g(y) = \text{ the } a \in X \text{ that satisfies } f(a) = y.$$
3. Then for all $x \in X$, we have $g(f(x)) = x$. ← see later in lecture.
4. Therefore g is a left inverse of f.
5. On the other hand for $y \in Y$, we have $f(g(y)) = y$.
6. Therefore g is a right inverse of f.
7. Therefore g is an inverse for f.

Solution to Problem 1

1) We're assuming this, because this is the assumption of the theorem.

2). When making a def, have to make sure it makes.

E.g. What goes wrong if I define

g: Humans → Humans

$g(x)$ = the child of x

Problems: ~~if~~ you plug in x, where x doesn't have a child, this doesn't make sense.

Problem 2 (10 points)

(a) Let $f : \{1, 2, 3\} \to \{1, 2, 3, 4\}$ be given by $f(x) = x$.

Define a left inverse to f. How many left inverses to f are there? How many right inverses to f are there?

(b) Suppose X and Y are nonempty sets. Prove that if $f : X \to Y$ is injective, then f has a left inverse.

(c) Suppose X and Y are nonempty sets. Prove that if $f : X \to Y$ has a left inverse, then f is injective.

Solution to Problem 2

(a)

(b) Look at Q1, and 2a) as inspiration.

(c)

Problem 3 (10 points)

Consider the following scenario.

Bob is leading his elementary school students on a field excursion. As they leave the bus, Bob counts 23 children. At the end of the day, as they pack into the bus to go home, Bob counts the number of children entering the bus. There are only 22, so Bob knows that a child is missing. See Lec 18

The purpose of this problem is for you to practice the skill of *capturing a piece of reasoning* by turning it into a theorem statement.

(a) Formulate and prove the theorem that Bob is using to deduce that there is a child missing. Explain why your choice is applicable. What is A? what is B.

~~Hint: When formulating the theorem, think in terms of subsets, not injections/bijections/surjections.~~

Hint: After you formulate the theorem, the cleanest way to prove it is probably to use the pigeonhole principle.

(b) Carl is a different teacher. He does not know how to count. However, he is still able to ensure that children are not left behind on field trips. What is a possible way for him to do this? Describe what he does as the children come back onto the bus.

Solution to Problem 3

(a)

(b)

For part a) Choose from the following alternatives:

Theorem if A, B are sets, $|A| < |B|$, then $B - A = \phi$.

Theorem if $A \subset B$ and $|A| = |B|$ then $A = B$.

Theorem if $A = B$ then $|A| = |B|$.

Problem 4 (10 points)

Prove this using pigeonhole principle, by defining an appropriate function from an appropriate domain to an appropriate codomain. See the writing sample.

If X is a subset of $\{1, 2, \ldots, 90\}$, and X has 46 elements, then X contains two numbers which are consecutive.

~~*No credit for using pigeons and pigeonholes.* Of course, the idea of pigeons and pigeonholes are lurking behind all of this, and you will need to cite the pigeonhole principle, but I want you to practice using the more formal language of injective~~ maps.

Solution to Problem 4

Problem 5 (10 points)

Consider the following statment.

If 8 distinct numbers are chosen from the set $\{1, 2, 3, \ldots, 13\}$, then there are two of these numbers which sum to to 14.

(Try it! Pick 8 random numbers and see if it works!)

(a) Is the statement still true if we change '8 distinct numbers' to '8 numbers'? Why not?

(b) Use the pigeonhole principle to prove the statement given in the setup (The '8 distinct numbers' version). **Hint:** if, when you read your proof, it looks like could also work for the '8 numbers' version, then you know you need to clarify some parts.

~~This time, I want you to use the language of pigeons and pigeonholes, rather than the language of injective functions~~.

Solution to Problem 5

(a)

(b)

Problem 6 (10 points)

Consider the following statement, and a proof. Recall that two sets are *disjoint* if they have no elements in common (i.e. their intersection is empty.)

Theorem: Let A be a set containing ten positive integers, each less than or equal to 100. Prove that there exist two *disjoint*, non-empty subsets of A which have the same sum of elements.

Example: If $A = \{1, 5, 7, 94, 32, 11, 3, 23, 4, 88\}$, then $23+88 = 111$ and $1+5+7+94+4 = 111$. You're asked to prove that no matter what set A is chosen, we can find something like this.
Proof:

domain/range needs to be changed, this is 2 mistakes

1. Let $A \subset \{1, 2, \ldots, 100\}$ have 10 elements.
2. Define the function $f : \mathcal{P}(A) \to \{1, \ldots, 500\}$ by

$$f(B) = \text{ the sum of the elements in the set } B.$$

3. Since $|\mathcal{P}(A)| = 2^{|A|} = 2^{10} = 1024$, which is greater than 500, we have the pigeonhole principle that f is not injective. Thus there are two distinct elements $B, B' \in \mathcal{P}(A)$ such that $f(B) = f(B')$.
4. By definition of f, B and B' are distinct subsets of A, which have the same sum, which is what we wanted to find.

(a) Check your understanding of the proof by assuming $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ in the first step and then writing down what $f(\{2, 3, 4\})$ and $f(\{9, 10, 12, 13\})$ is.

(b) There are at least 3 gaps/mistakes in this proof. What are the 3 most major mistakes/gaps? Plug the gap(s) and/or fix the mistake(s). ~~Is the theorem even true~~? *Note: In the proof, we used that $|\mathcal{P}(A)| = 2^{|A|}$. This is a true fact that we will prove later.*

The theorem is true. Plug the gaps.

Solution to Problem 6

(a)
- $f(\{2, 3, 4\}) =$
- $f(\{9, 10, 12, 13\}) =$

(b)
(a) The first issue is that...
(b) The second issue is that...
(c) The third issue is that...

Problem 7 (10 points)

This is an optional extra credit problem. It is harder than the other problems (but the solution is still quite short).

For $x \geq 0$, let $F(x)$ denote the non-integer part of x. More formally, $F(x)$ is the unique number in $[0, 1)$ such that $x - F(x)$ is an integer.

Examples: $F(4) = 0$, $F(3.999) = 0.999$, $F(3.5) = 0.5$, $F(5234.023) = 0.023$ and $F(\sqrt{2}) = 0.414213\ldots$.

If $F(x)$ is very small, it means that x is very close to an integer.

Prove that there is an integer $n > 0$ such that $F(n \cdot \sqrt{2}) < 0.00000001$.

Solution to Problem 7

Hint: Use pigeonhole principle.

Back to counting (Ch 10, 11).

We know

* What is counting
* Why do we count

Knowing $|A|$ and $|B|$ tells you whether ~~there~~ are injections/surjections/bijections $f: A \to B$

Today:

* Techniques for countings.

Theorem

if X and Y are disjoint sets,

$$|X \cup Y| = |X| + |Y|.$$

Proof: Suppose $|X| = n$ and $|Y| = m$

Let $f: \mathbb{N}_n \to X$ and $g: \mathbb{N}_m \to Y$ be bijections, Need: $|X \cup Y| = n+m$.

which means we need

bijection

$h: \mathbb{N}_{n+m} \longrightarrow X \cup Y$.

See textbook, will need induction

Theorem (Principle of inclusion/exclusion).

If X and Y are sets

$$|X \cup Y| = |X| + |Y| - |X \cap Y|$$

Example

$X = \{2, 3, 4\}$

$Y = \{4, 5, 6\}$

How many elements in union? $\{2, 3, 4, 5, 6\}$.

$|X| = 3$

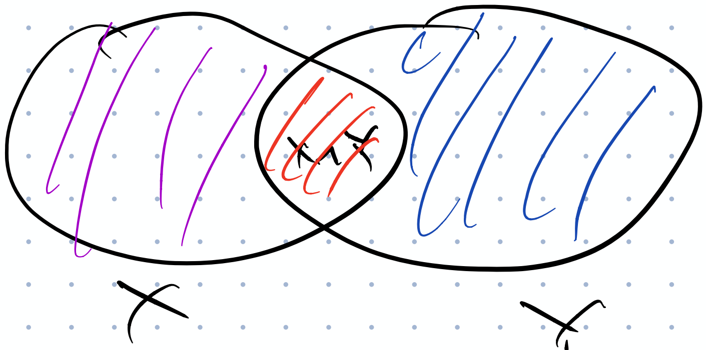
$|Y| = 3$

$|X \cap Y| = 1$

So $|X \cup Y| = 3 + 3 - 1 = 5$.

Proof

First: $X-Y$, $X\cap Y$, $Y-X$



These are disjoint

and $(X-Y)\cup(X\cap Y)\cup(Y-X)=X\cup Y$

Using previous thm, because the 3 sets are disjoint.

$$|(X-Y)\cup(X\cap Y)\cup(Y-X)| = |X-Y| + |X\cap Y| + |Y-X|$$

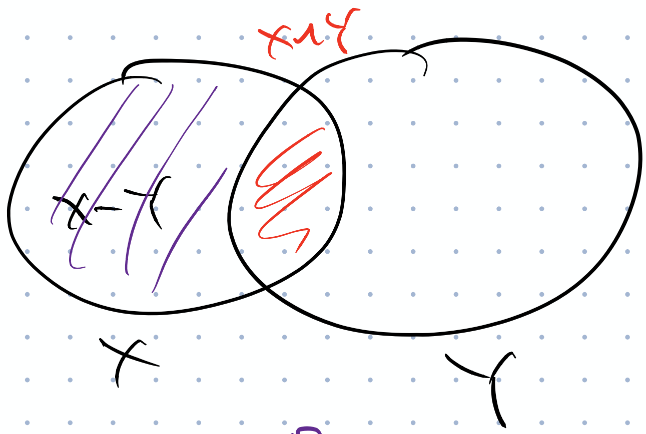
$$|X\cup Y| = |X-Y| + |X\cap Y| + |Y-X|$$

$$= |X|-|X\cap Y| + |X\cap Y| + |Y|-|X\cap Y|$$

$$= |X| + |Y| - |X\cap Y|.$$

Why is $|X-Y| = |X| - |X\cap Y|$?

Proof:



$X-Y$ and $X\cap Y$ are disjoint.

So

$$|\underline{(X-Y)\cup(X\cap Y)}| \overset{\text{By Prev. Thm}}{=} |X-Y| + |X\cap Y|$$

$$|\underline{X}| = |X-Y| + |X\cap Y|$$

So $|X-Y| = |X| - |X\cap Y|$. ∎

We technically didn't use the prev. Theorem:

We actually used.

Theorem: if X, Y, Z are disjoint sets,

$$|X\cup Y\cup Z| = |X| + |Y| + |Z|.$$

To prove this, we would use prev. Theorem.

Then, you would prove:

Theorem:

if $X_1, \ldots, X_n$ are disjoint sets,

$$|X_1 \cup \cdots \cup X_n| = |X_1| + \cdots + |X_n|.$$

Proof: by induction. (on n).

Principle of Multiplication:

$|X \times Y|$

Reminder:

$X = \{a, b, c\}$

$Y = \{0, 1\}$

$$X \times Y = \{(a,0), (a,1), (b,0), (b,1), (c,0), (c,1)\}.$$

Theorem

if A, B are sets,

$$|A \times B| = |A| \times |B|.$$

$$X \times Y = \{ (x,y) : x \in X, \ y \in Y \}.$$

$$X^2 = \{ (x,y) : x \in X, \ y \in X \}.$$

$$X = \{0,1\}$$

$$X^2 = \{ (0,0), (0,1), (1,0), (1,1) \}$$

$$\neq \{ (0,0), (1,1) \}$$

$$= \{ (x,\cancel{y}) : x \in X, \ \cancel{y} \in X \}.$$

So $X^n = \{ (x_1, \dots, x_n) : x_i \in X \}.$

<u>Theorem</u>

$$|X^n| = |X|^n.$$

E.g. There are 3 possible pant sizes: $\{S, M, L\}$.
6 possible pant colors: $\{R, O, Y, B, G, V\}$.

Q: How many pant configurations.

Ans:

$$C = S \times \text{Colors}$$

So by theorem.

$$|C| = |S| \times |\text{Colors}|$$

$$= 18$$