Last time: formalized what it means to count a set.

Let X be a set

$|X| = n$ means there is a bijection $f: \mathbb{N}_n \to X$.

On HW6 Q8, I asked <u>why</u> we count.

Most answers missed the point.

wrong question:

* What is counting
  How to define counting?

Another type of response:

* Counting is useful because it allows to know how many things you have.

This is circular.

Why is it important

it allows to know how many things you have.

* Counting helps us understand what a bijection is.

* Counting is <u>obviously</u> important in real life.

* Counting is useful for calculus/ computer science.

<u>Getting closer</u>

Counting can be used to figure out if we can <u>divide</u> apples for children.

E.g. if 12 apples, 4 children, each child can get 3 apples.

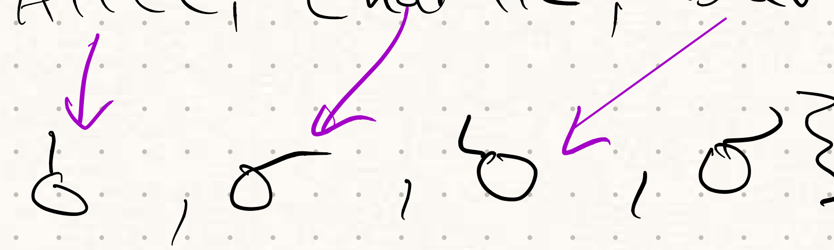
<u>Best example</u>

if 4 apples, 3 children, each child can get their own apple.

<u>Best answer:</u>

Counting allows us to quickly determine whether there is an injection, surjection or bijection between 2 sets.

$C = \{\text{Alice}, \text{Charlie}, \text{Dave}\}.$

$A = \{$  $\}.$

Since $|C| = 3$, $|A| = 4$, we know there is an injective $f: C \to A$.

(Do something HW7 P3).

Theorem If $|A| \leq |B|$ then there is an injection

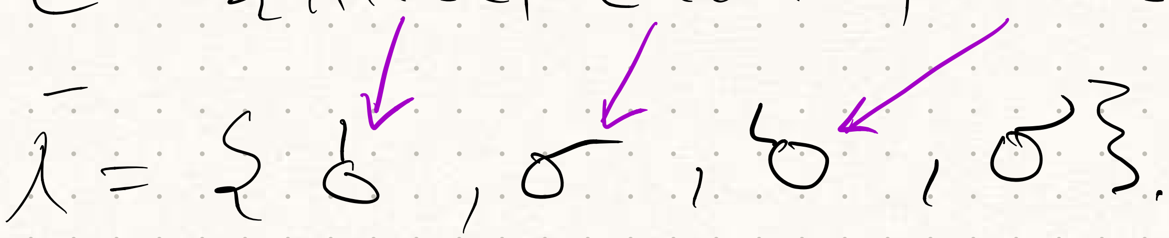
$$f : A \to B.$$

Proof: Hints:

* First reduce to the case $A = \mathbb{N}_n$, $B = \mathbb{N}_m$.
* Use induction.

Can every child get their own apple?

$C = \{\text{Alice}, \text{Charlie}, \text{Dave}\}.$



Can you do this if you don't know how to count?

Yes, just start drawing arrows / giving out apples.

Pigeonhole principle: (PHP)

if you have n pigeonhole,

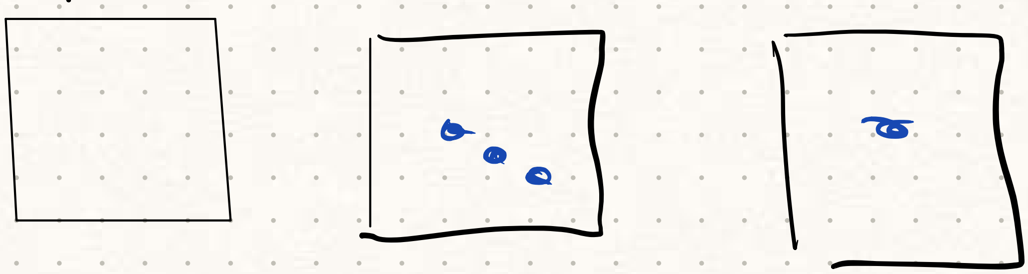
m pigeons

if $m > n$, putting all pigeons in pigeonholes

there is one

pigeonhole with 2 pigeons.

Example.



4 pigeons

3 pigeonholes,

at least one pigeonhole has ≥ 2 pigeons.

Theorem (PHP)

if $|A| > |B|$, and $f: A \to B$ is a function,

then f is not injective.

Example application

Fact: There are 2 people in NYC with the same number of hairs on their head.

Proof:

Let $A = \{\text{people in NYC}\}$

$B = \{1, 2, ---, 10^6\}$

Then $|A| > |B|$.

Let $f: A \to B$

$f(x) =$ number of hairs on x's head.

By PHP, f is not injective.

So $\exists\, x \neq y$, $f(x) = f(y)$

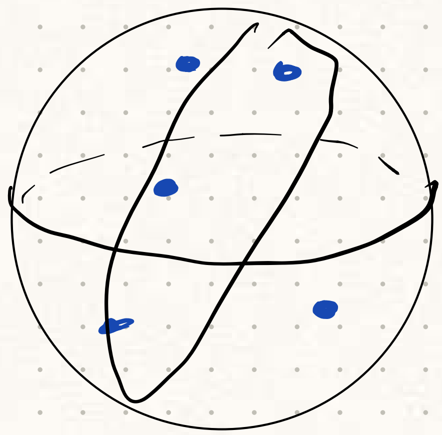
hairs on x's head $=$ # hairs on y's head

$\blacksquare$

Note: in the proof: f is well defined because humans have $\sim 10^5$ hairs on head.

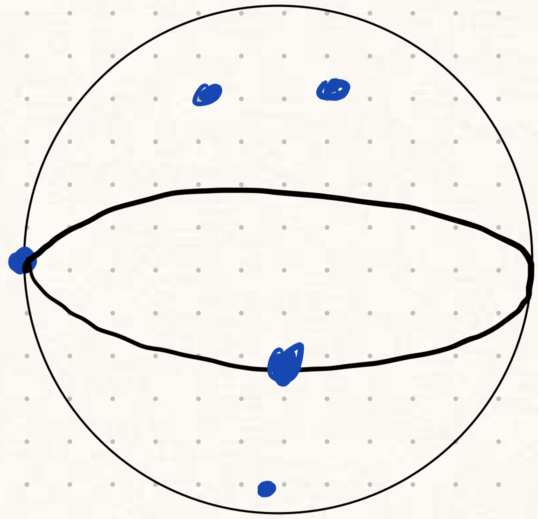
Example Application

Let A be a set of 5 points on 2-sphere.



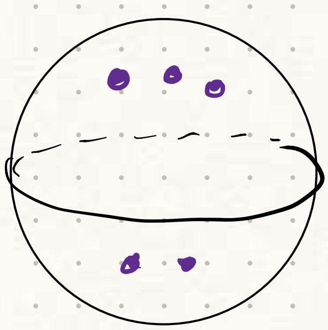
Then there is a closed hemisphere that contains 4 of the points.

Proof: Pick any 2 points in A, and rotate the sphere so that they lie on the equator.



There are 3 points remaining, one of the hemispheres must contain 2 points. (by PHP)

So this hemisphere has 4 points. ∎



Example application:

If 6 distinct numbers are chosen from $\{1, 2, \ldots, 9\}$.

Then 2 of them sum to 10.

Example

7 2 1 3 4 5

Proof:

Let $A =$ the set of 6 numbers that were chosen from $\{1, 2, \ldots, 9\}$.

Let $B = \{\{1,9\}, \{2,8\}, \{3,7\}, \{4,6\}, \{5\}\}$

Then $|A| = 6$

$|B| = 5$.

Let $f : A \to B$

$f(x) =$ the set in B that contains x.

Example: $A = \{7, 2, 1, 3, 4, 5\}$.

$f(2) = \{2, 8\}$.

$f(7) = \{3, 7\}$.

$f(5) = \{5\}$

$f: A \to B$ is not injective (by PHP).

so there is $x, y \in A$, $x \neq y$, $f(x) = f(y)$.

the set in B that contains x.

$=$

the set in B that contains y.

---

x, y are in the set in B.

So $x + y = 10$ (because we chose the sets cleverly).

As we see, applying PHP can be tricky.

To prove Thm using PHP.

Need to:

1) Pick A

2) Pick B

3) Pick $f: A \to B$

4) PHP tells you (if $|A| > |B|$) f is not injective.

Now you need to explain why this helps you prove thm.

Proof:

Let $A =$ the set of 6 numbers that were chosen from $\{1, 2, \ldots, 9\}$

Let $B = \{\{1,2\}, \{3,4\}, \{5,6\}, \{7,8\}, \{9,10\}\}$

To prove Thm using PHP.

Need to:

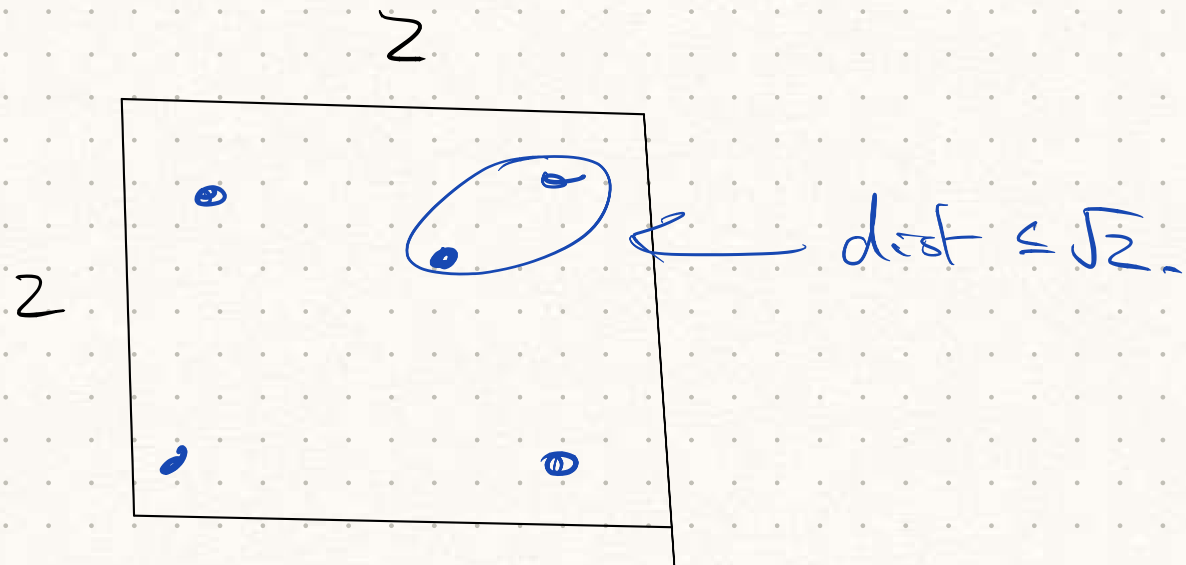
1) Pick Pigeons
2) Pick Pigeonholes
3) Pick way to assign pigeonholes
4) PHP tells you (if $|A| > |B|$) one of the pigeonholes has 2 pigeons.

Now you need to explain why this helps you prove th

Example

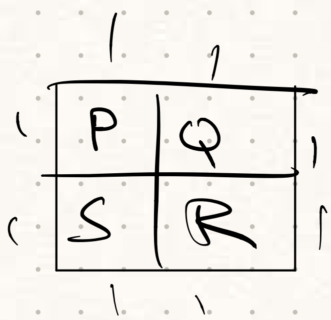
Prove that if 5 points are chosen in a square of side length 2.

then 2 of them are with $\sqrt{2}$ of each other.

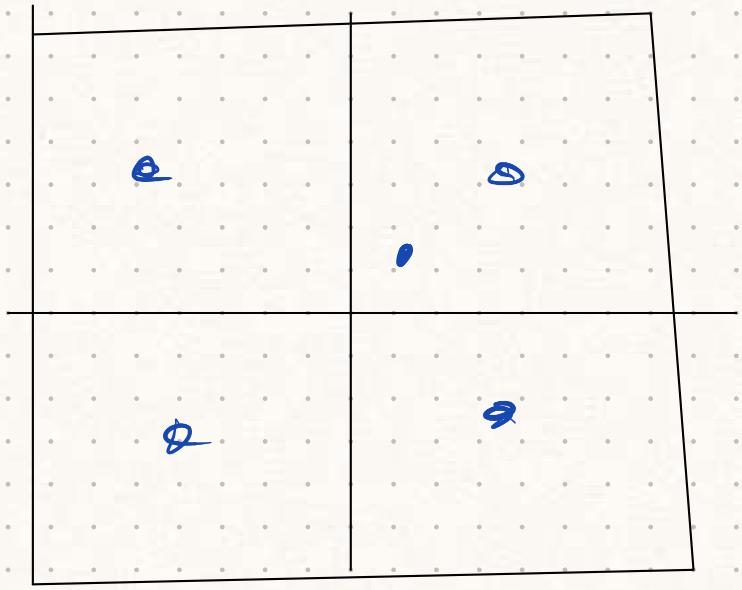


Proof: Pigeons: the 5 points.

Pigeonholes: $\{P, Q, S, R\}$.



Put x in the sub square containing x



By PHP, 2 pts must be in the same unit subsquare.

So these points must be less than $\sqrt{2}$ apart. $\boxtimes$