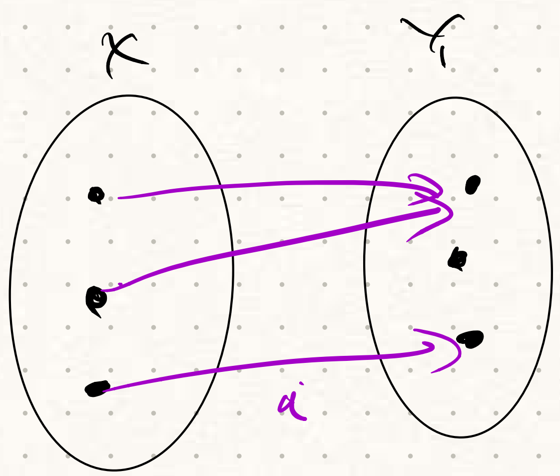
Lecture 17 (Ch 10, 11)

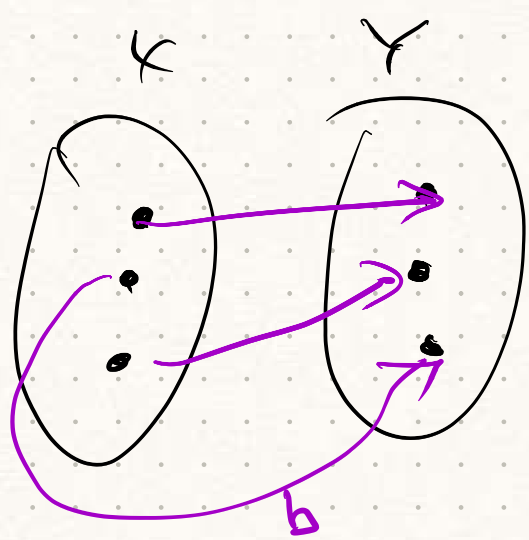
Last 3 lectures:

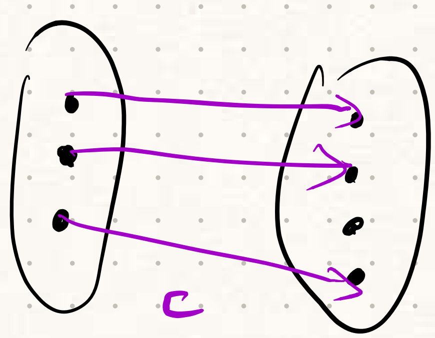
* Image, Pre-image
* Surjective, Injective, Bijective.
* Function composition
* Left/right inverse, inverse, invertible.

Recall: $f: X \to Y$ is a bijection means f is surjective and injective.

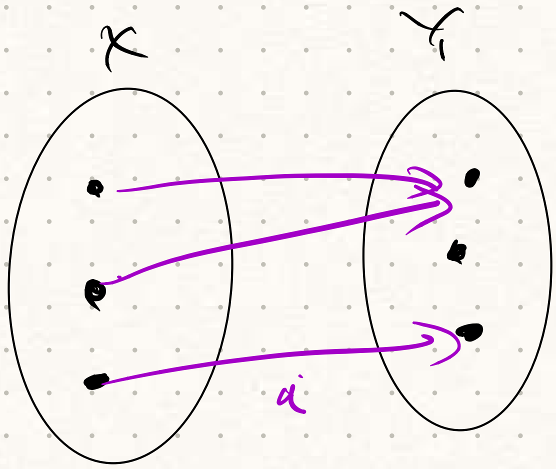
Quiz:







Which ones are bijections



not injective.

not bijective.

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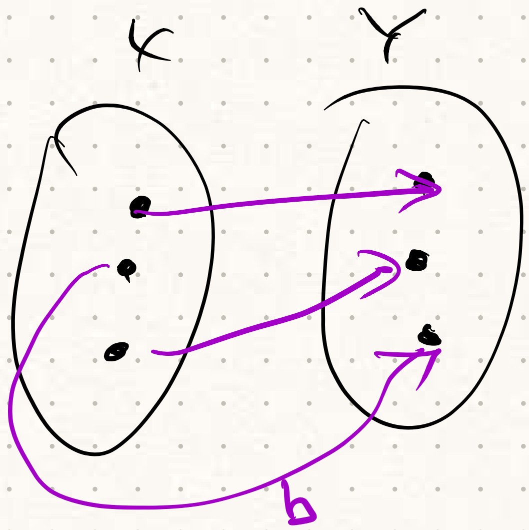
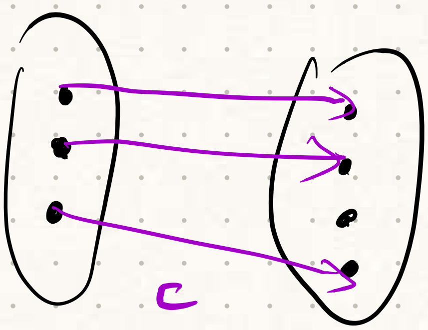
injective ✓

surjective ✓

bijective ✓





not surjective.

not bijective.

Theorem f is invertible

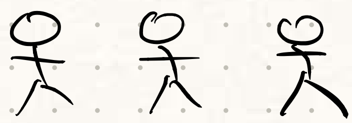
$$\Longleftrightarrow$$

f is bijective

This will be a homework problem. I will write down the proof and ask you to justify each step.

Today: Counting (Ch 10, 11)

Example



There are 3 people here.

$\{2, 4, 7\}$ There are 3 elements in this set.

What do these statements really mean?

What is counting?

Definition

$$\mathbb{N}_n := \{1, 2, 3, \ldots\ldots, n\}$$

E.g. $\mathbb{N}_2 = \{1, 2\}$

$\mathbb{N}_7 = \{1, 2, 3, 4, 5, 6, 7\}$

Definition 10.1.1 Let X be a set. We say 'X has n elements' if 'there exists a bijection

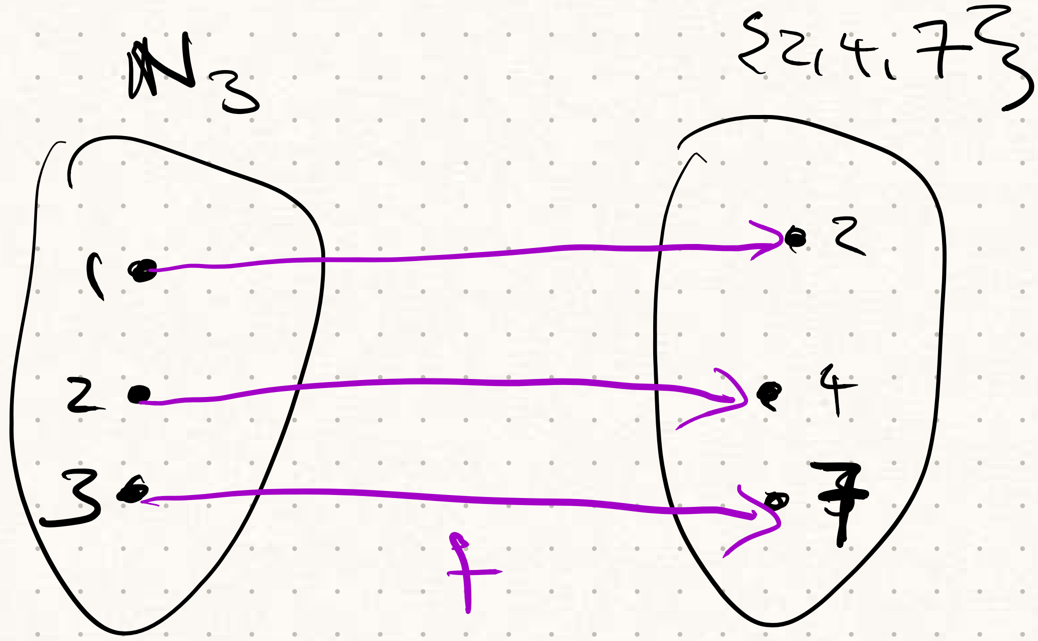
$$f : \mathbb{N}_n \longrightarrow X.'$$

Let's work out some examples to see why this formal definition matches our intuition of counting.

Example Prove that $\{2, 4, 7\}$ has 3 elements, using the definition.

Proof: Need to show there is a bijection $\mathbb{N}_3 \to \{2,4,3\}$.

Picture:



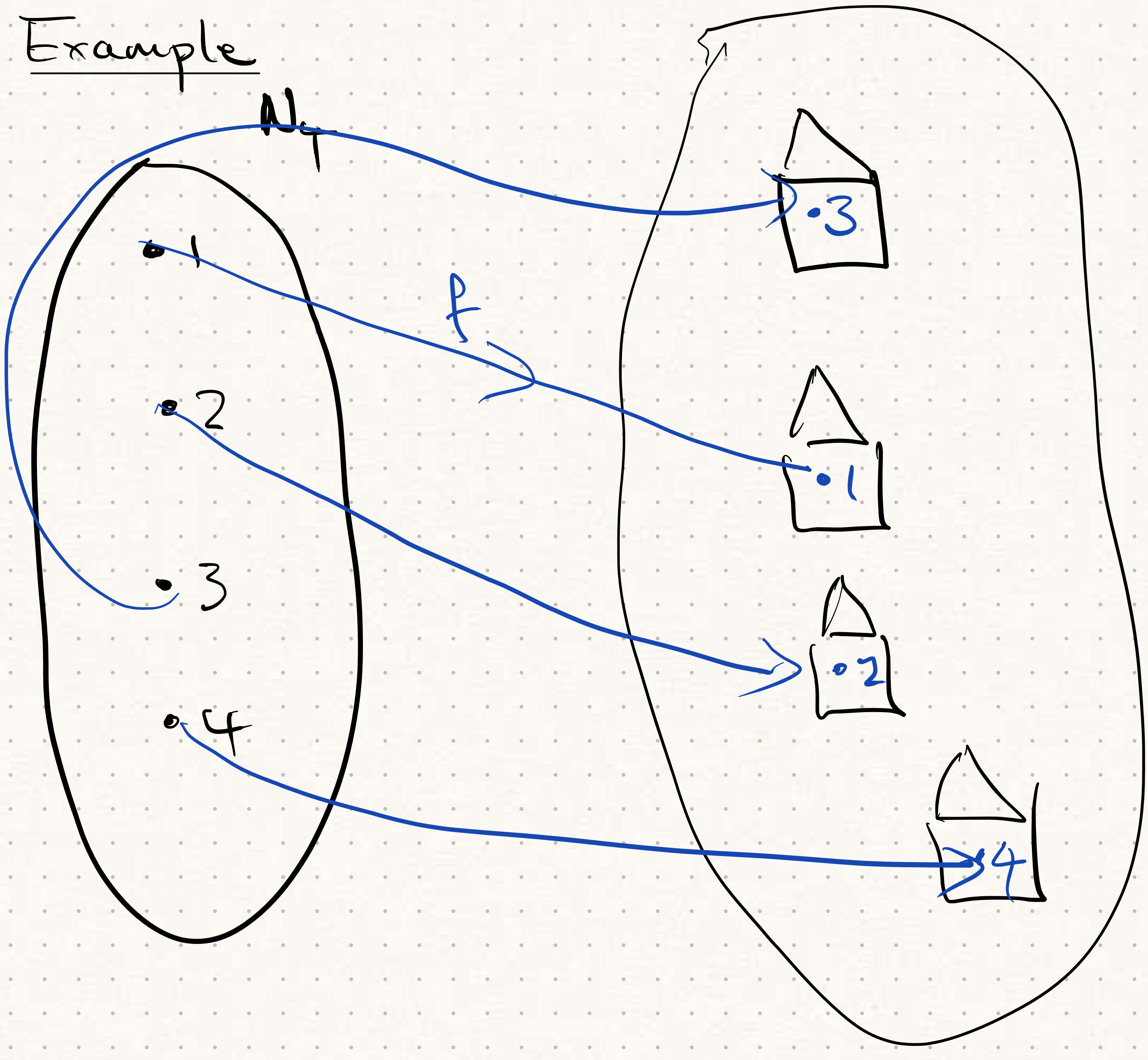
f is a bijection. $\blacksquare$

This example shows that the Def^n is matching with our intuition.

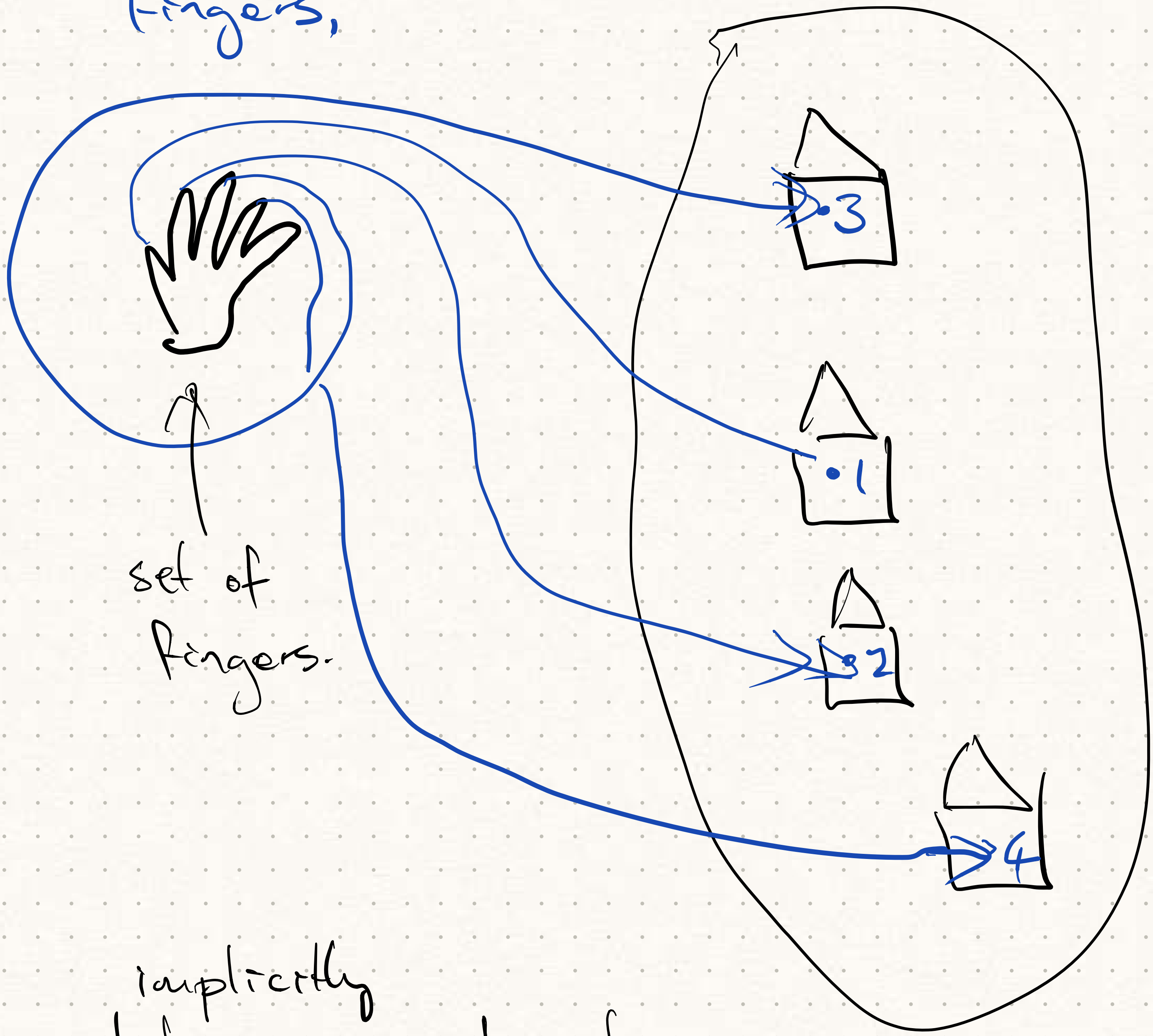
In fact,

* every time you count aloud, you are implicitly defining a bijection
$f: \mathbb{N}_n \to$ the things are counting.

<u>Example</u>



* Every time you count with fingers,



implicitly defining a bijection

f: Fingers $\to$ the set.

Examples Here are some common mistakes that everyone makes when counting.

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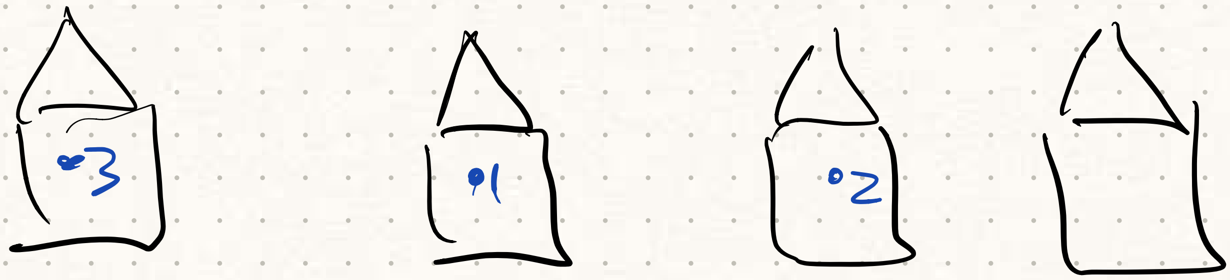
a)

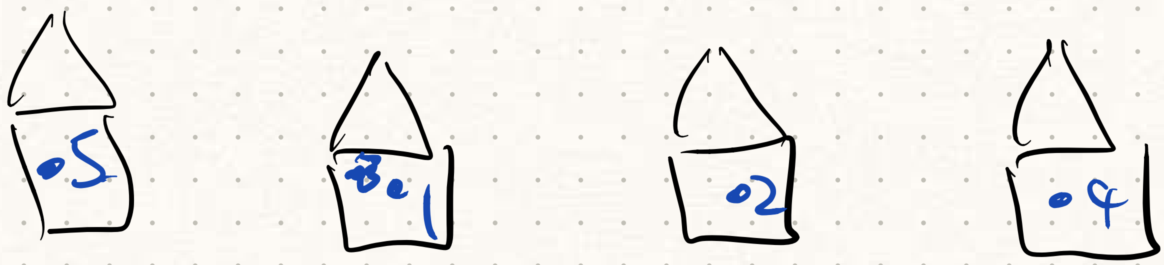


There are 3 houses.

What went wrong?

The function I have implicity defined is not surjective, so not bijective.

b)



There are 5 houses.

What went wrong?

The function I have implicity defined is not injective, so not bijective

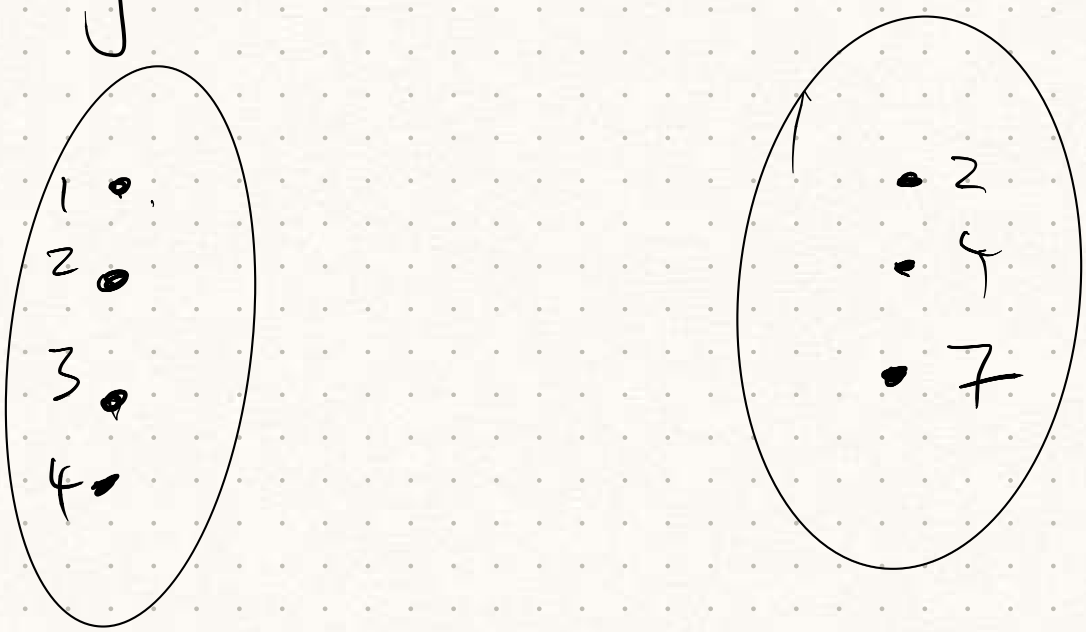
Ambiguity?

We proved that $\{2,4,7\}$ has 3 elements.

But we need to make sure its unambiguous.

What if $\{2,4,7\}$ also has 4 elements?

I.e. what if there was a bijection $f: \mathbb{N}_4 \to \{2,4,7\}$?



Lesson: For the definition to be good, we need:

Theorem if $f: \mathbb{N}_m \to X$ and $g: \mathbb{N}_n \to X$ are bijections, then $n=m$.

Proof: HW.

Summary:

When making a definition

* Explain why it is a "useful" definition

* Make sure its unambiguous.

Question: Is there a bijection $f: \phi \to \phi$.

Yes, $f = \{\ \}$

E.g. $f(x) = x^2$, $f: \mathbb{R} \to \mathbb{R}$

$\{(1,1), (2,4), (-2,4), \ldots\}$

$f = \{\}$ is surjective and injective

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surjective: $f: X \to Y$

$\forall y \in Y \ \exists x \in X \ f(x) = y$

surjective: $f: \emptyset \to \emptyset$

$\underbrace{\forall y \in \emptyset \ \exists x \in \emptyset \ f(x) = y}_{\text{Vacuously true.}}$

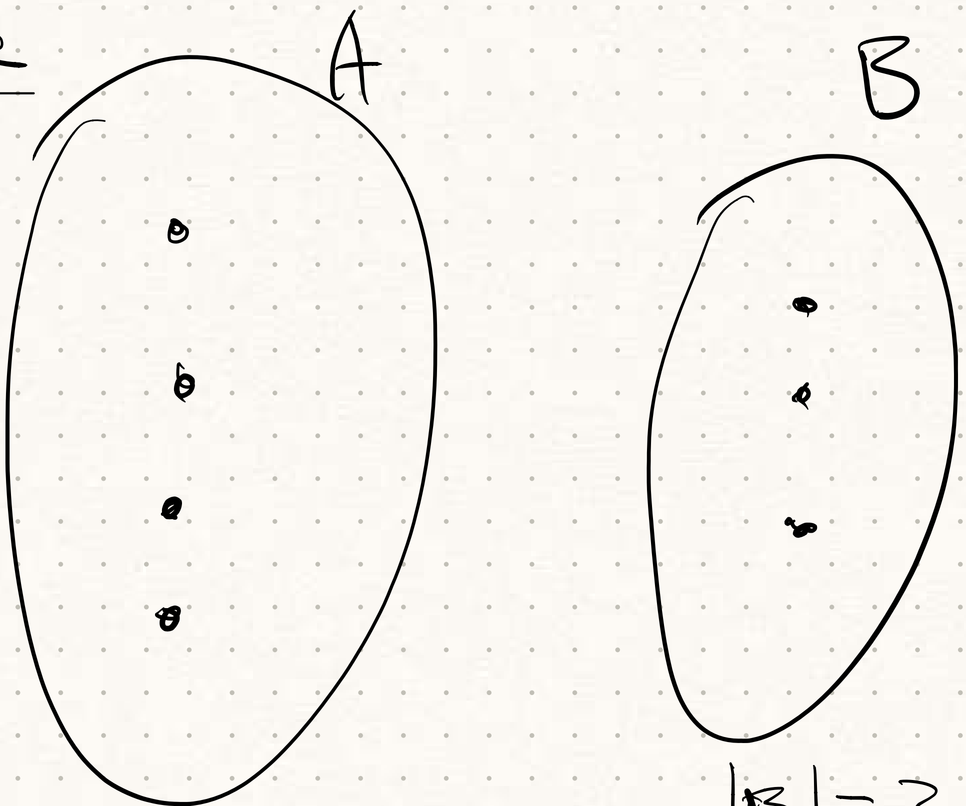
Definition

$|X|$ is the number of elements in X

Theorem Pigeonhole principle

if $|A| > |B|$ then there are no injective functions $A \to B$.

Example



$|A| = 4$

$|B| = 3$

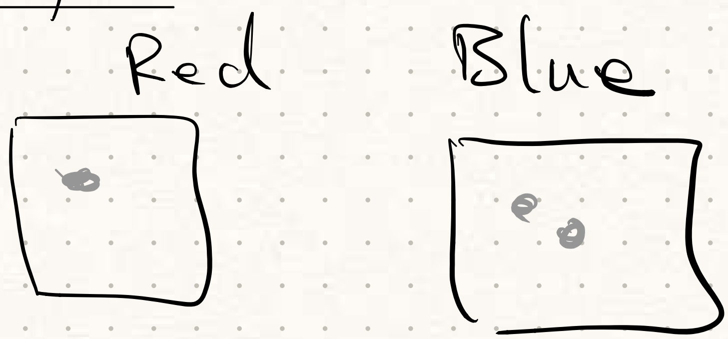
$|A| > |B|$, no injective funcs

Applications of PHP:

* Suppose you a drawer with red and blue socks.

Prove that if you pull out 3 socks, you are guaranteed to have a matching pair.

Informal Proof:



Proof

Let $f: \{1,2,3\} \to \{red, blue\}$

$f(i) =$ the color of the i'th sock

Since $|\{1,2,3\}| = 3$

$|\{red, blue\}| = 2$, and $3 > 2$,

f cannot be injective. (by PHP).

f cannot be injective,

so there is $i \neq j \in \{1,2,3\}$

such $f(i) = f(j)$

so

the color of i'th sock = the color of j'th sock

$\boxtimes$

Next time: more applications for PHP.

$f: \mathbb{R} \to \mathbb{R} \times \mathbb{R}.$

$$\mathbb{R} \times \mathbb{R} = \{(a,b) : a \in \mathbb{R},\ b \in \mathbb{R}\}$$
$$= \mathbb{R}^2$$

$f:$ US Citizens $\to \mathbb{Z}$

$$\mathbb{Z} = \{-1, 0, 1, 2, -2, 3, -3, \ldots\}$$

13:

stateOf(11776) = NY

stateOf(11772) = NY.

$$f : \mathbb{R} \to \mathbb{R}_{\geq 0}$$
$$f(x) = -x^2$$

f is not well-defined.