Lecture 16 Inverses and bijections (Ch 9.)

(HW Due Thursday)

Last time.

* $f : X \to Y$ is a bijection

$\iff$

f is surjective and injective.

* $g : Y \to Z$ then

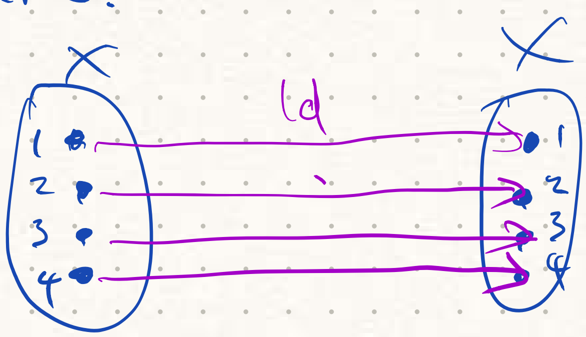
$g \circ f$ is a new function,

$$g \circ f : X \to Z.$$

* $Id_X : X \to X$ is the function

$$Id_X(a) = a \quad \text{for } a \in X$$

Picture:



* $g: Y \to X$ is a left inverse to $f: X \to Y$ if

$g \circ f = \mathrm{Id}_X$. $\to$ i.e. For all $a \in X$ $g(f(a)) = a$

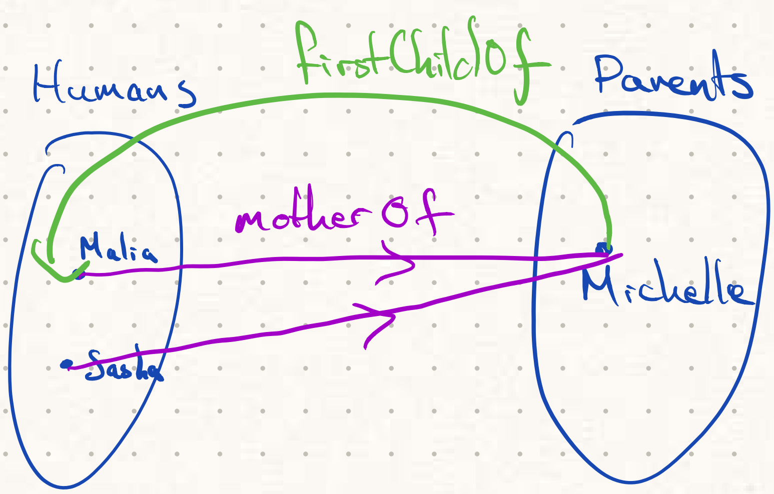
Example firstChildOf : Parent $\to$ Human

motherOf : Humans $\to$ Parents

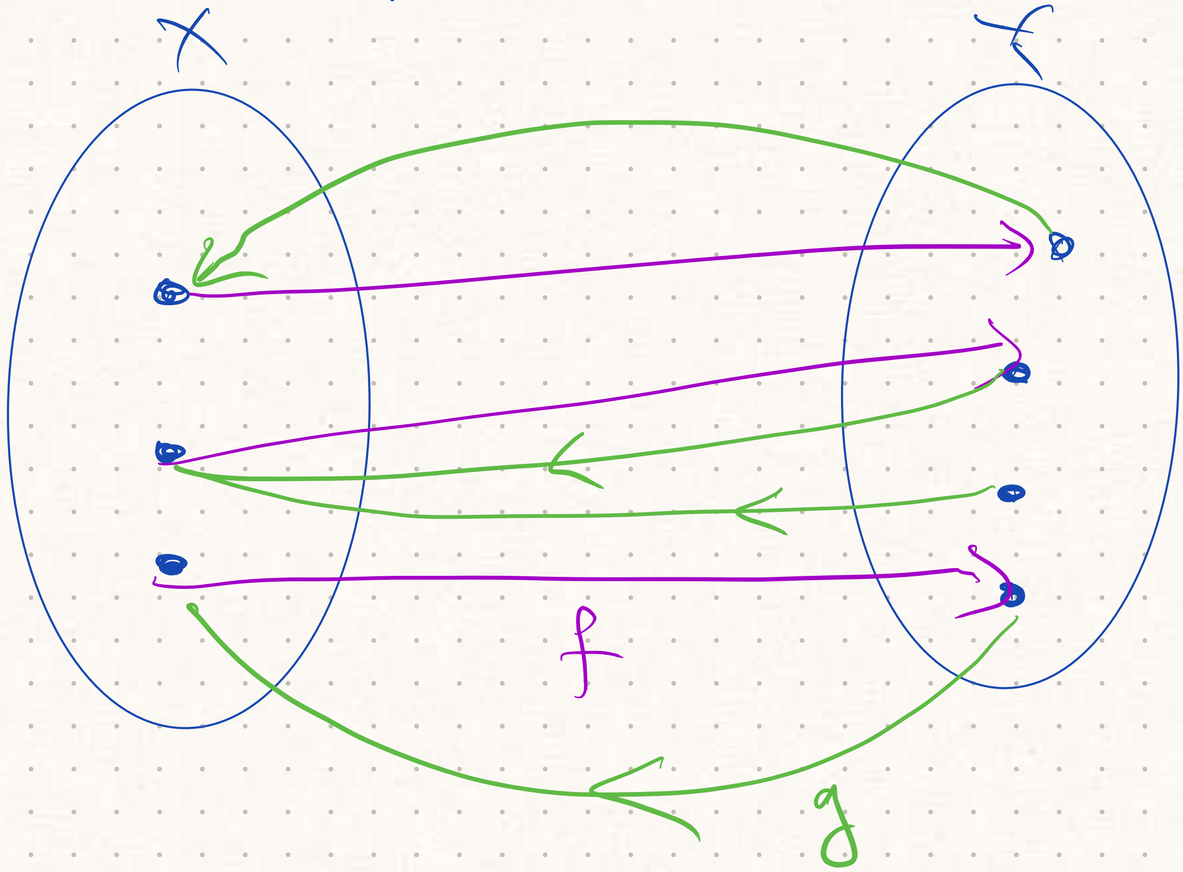
firstChildOf is <u>not</u> left inverse of motherOf, because

firstChildOf (motherOf (Sasha))

= firstChildOf (Michelle) = Malia



Pictoral interpretation of left inverse:

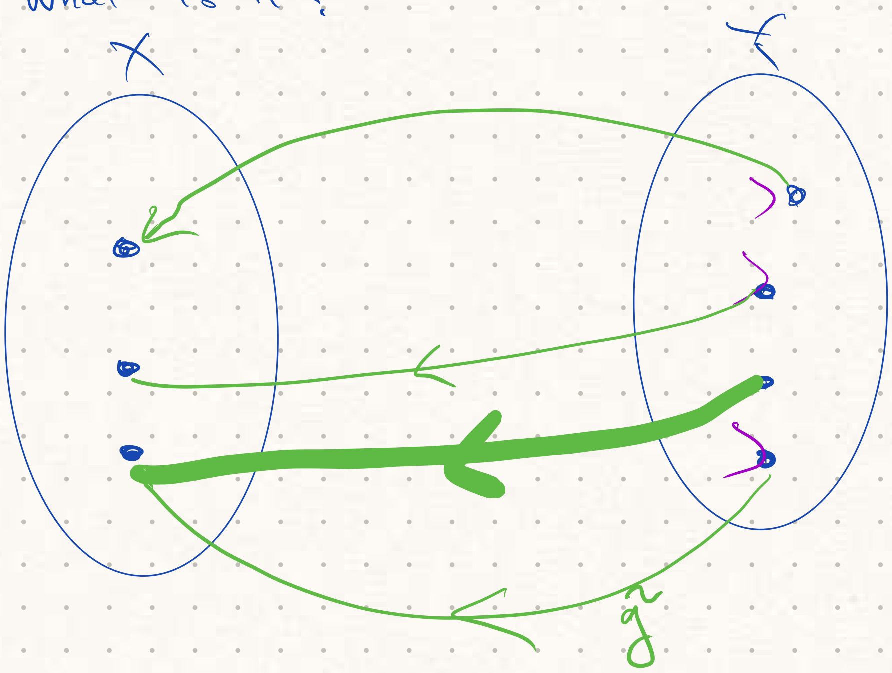


$g \circ f = \mathrm{Id}_X$, so g is a left inverse to f.

Excercise

There is a different left inverse for f. What is it?

Answer:



$\tilde{g} \circ f = \mathrm{Id}_X$, so $\tilde{g}$ is another left inverse of f.

Another example:

$f: \mathbb{R} \to \mathbb{R}_{\geq 0}, \quad f(x) = x^2$

$g: \mathbb{R}_{\geq 0} \to \mathbb{R}, \quad g(x) = +\sqrt{x}$

Is f a left inverse for g?

66% True

Answer:

Want to check if

$$f \circ g = \mathrm{Id}_{\mathbb{R}_{\geq 0}} \quad ✓$$

i.e.

$$f(g(x)) = x \qquad \text{for } x \in \mathbb{R}_{\geq 0}$$

$$(+\sqrt{x})^2 = x \quad ✓$$

So f <u>is</u> a left inverse for g.

Is g a left inverse for f?

66% No

Answer: No.

$$g(f(x)) = x \quad ?$$

$$+\sqrt{x^2} = x \quad ?$$

Not true, take $x = -1$.

Excercise

Is $\text{motherOf}: \text{Humans} \to \text{Parents}$ a left inverse of

$\text{firstChildOf}: \text{Parents} \to \text{Humans}$?

Answer: No.

Excercise

Is $\text{motherOf}: \text{Humans} \to \text{Mothers}$ a left inverse of

$\text{firstChildOf}: \text{Mothers} \to \text{Humans}$?

Answer: Yes.

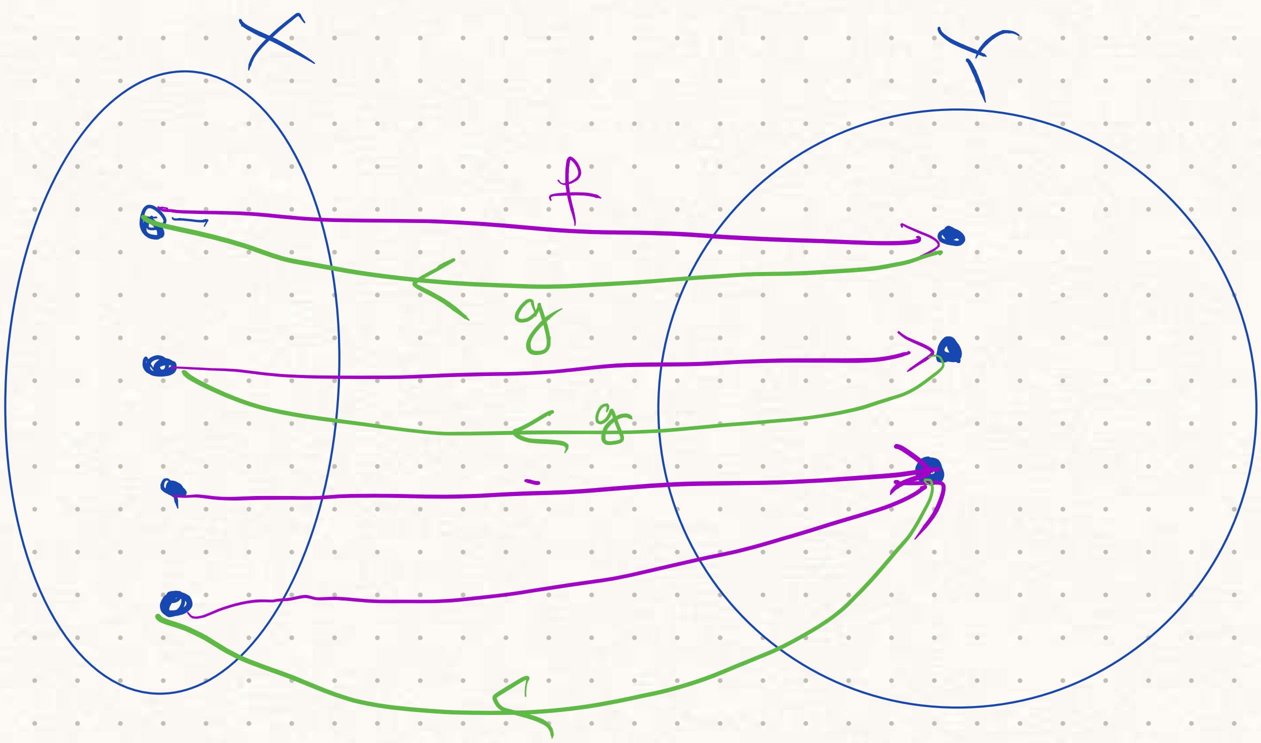
Definition

$g: Y \to X$ is right inverse of $f: X \to Y$ if

$$f \circ g = \mathrm{Id}_Y.$$

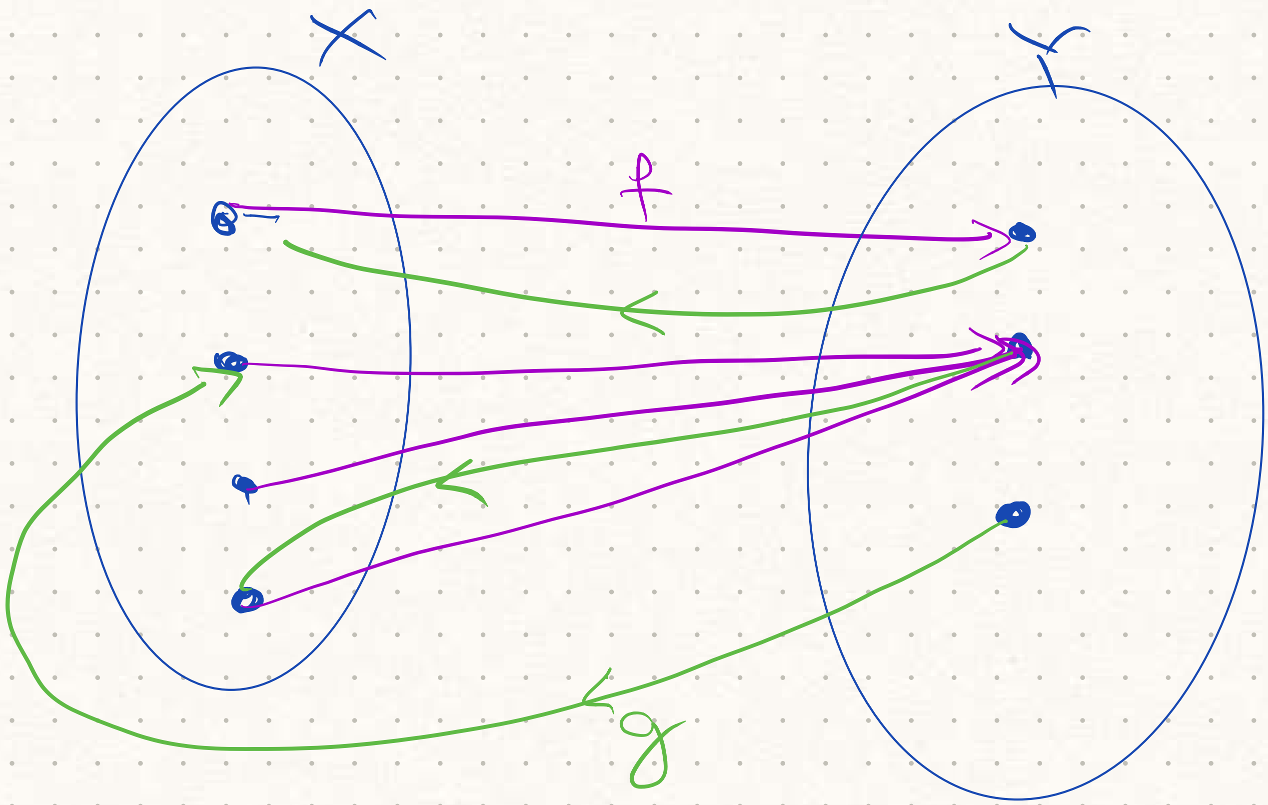
Excercise: Replace left with right, in previous examples.

Picture of right inverse



$f \circ g = \mathrm{Id}_Y$, so g is a right inverse for f.

Consider the modified f:



Find a right inverse for f.

This is impossible because f is not surjective.

<u>Theorem</u> If f is not surjective, f does not have a right inverse

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<u>Theorem</u> If f is not injective, f does not have a left inverse.

Theorem If f has a right inverse f is surjective.

Theorem If f has a left inverse then f is injective.

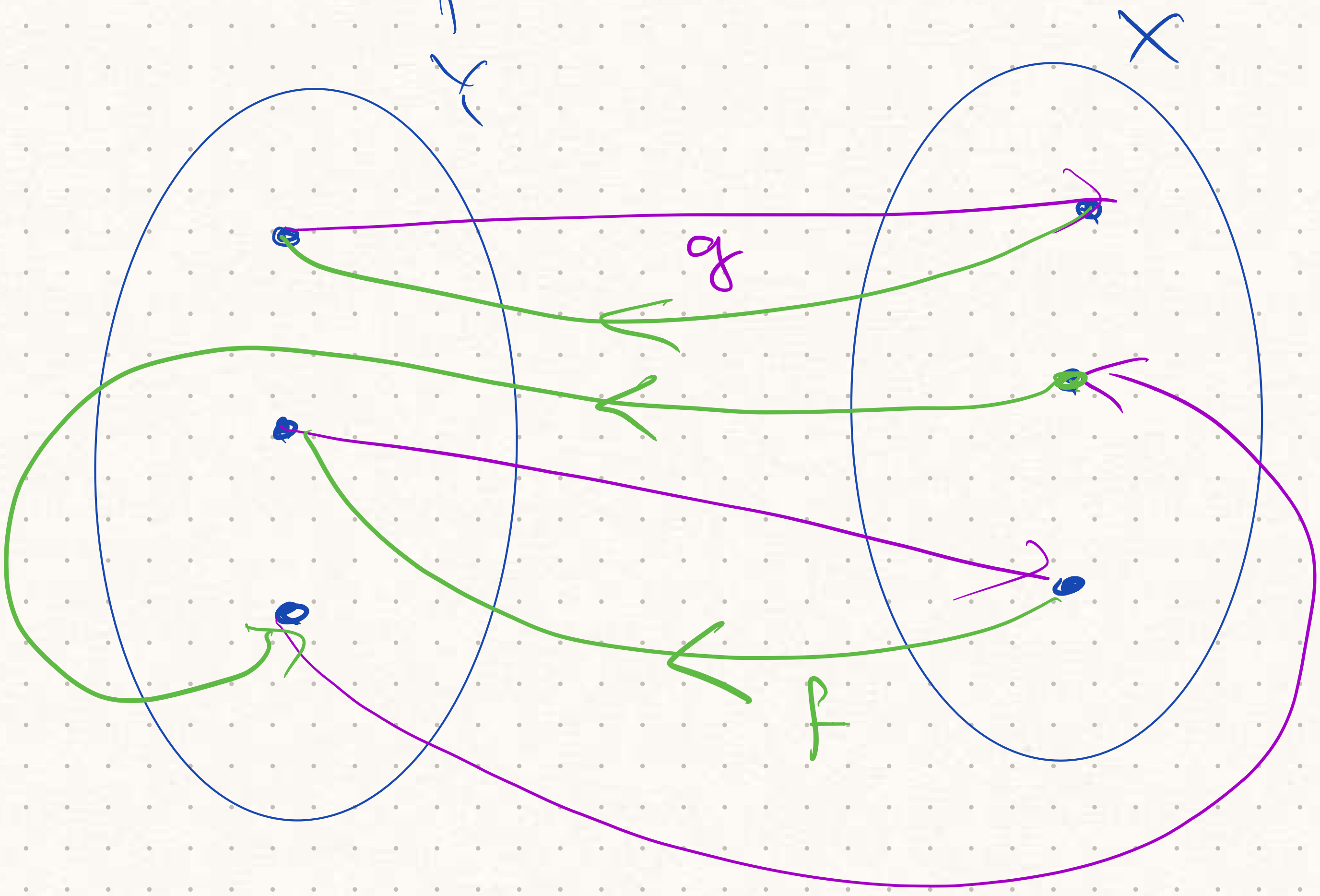
Definition $f: X \to Y$ is an inverse for $g: \cancel{X} \to X$ if (f is a left inverse for g) and (f is a right inverse for g).

In other words,

$$f(g(y)) = y \qquad g(f(x)) = x$$

for all $y \in Y$ and $x \in X$.

Pictoral interpretation:



Are there any other inverses to g?
No.

<u>Theorem</u>: if $g: Y \to X$ has an inverse, it has a unique inverse.

<u>Proof</u>: Let $f: X \to Y$ be an inverse to g. Let $\tilde{f}: X \to Y$ be another inverse to g. Goal: show that $f = \tilde{f}$.

First, note that

$$g \circ f = g \circ \tilde{f}$$

(both are equal to Id_X). because $f, \tilde{f}$ are inverse to g.

So

$$g(f(x)) = g(\tilde{f}(x)) \quad \text{for all } x \in X$$

by definition of function composition and function equality.

Apply f to both sides:

$$f\big(g(\underbrace{f(x)}_{a})\big) = f\big(g(\underbrace{\tilde{f}(x)}_{a})\big) \quad \text{for all } x \in X$$

So

$$f(x) = \tilde{f}(x) \quad \text{for all } x \in X$$

Because $f(g(a)) = a$ for all a.

So $f = \tilde{f}$. $\blacksquare$

Examples:

* The inverse of
$$\exp : \mathbb{R} \to (0, \infty)$$
is
$$\log : (0, \infty) \to \mathbb{R}$$

* The inverse of
$$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$$
is $\arcsin : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Inverses and bijections.

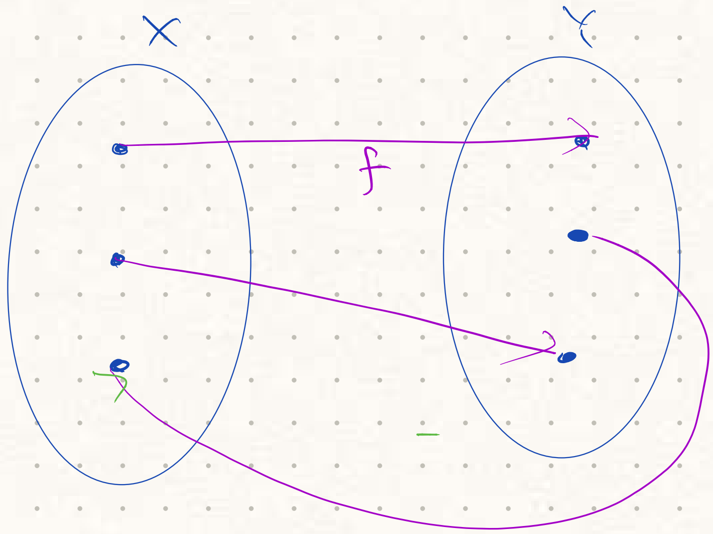
Definition $f : X \to Y$ is called invertible if it has an inverse.

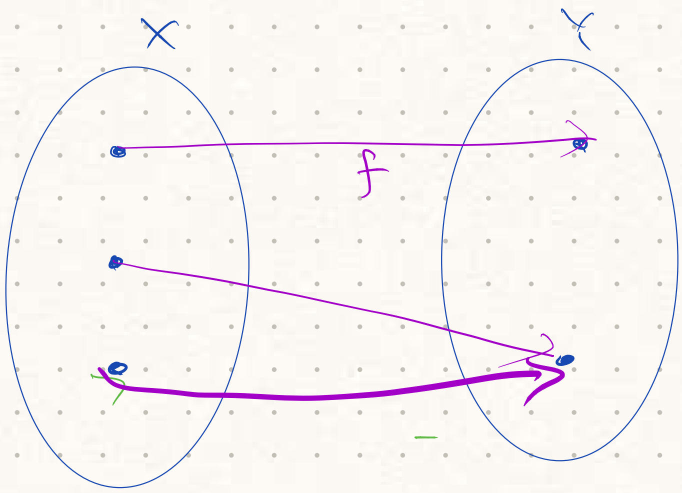
Excercise Show that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is not invertible.

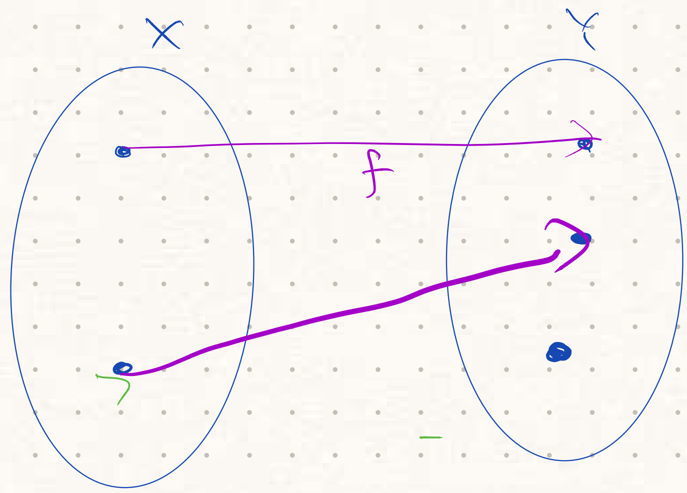
Theorem f is invertible
$$\Longleftrightarrow$$
f is a bijection.

Consequently, to prove that f is a bijection, suffices to find an inverse for f.

(See HW Problem 4 and 5).







Excercise: Which ones are bijective?
Which ones have inverses?