Lecture 25

Last time: image, pre-image, injective, surjective.

$f: X \to Y$

* image: $\operatorname{im}(f) = \{f(x) : x \in X\}$

* pre-image: For $y \in Y$

$$f^{\leftarrow}(y) = \{x \in X : f(x) = y\}$$

* injective: means

"For all $y \in Y$ $f^{\leftarrow}(y)$ has at most one element"

* surjective:

"For all $y \in Y$ $f^{\leftarrow}(y)$ has at one element"

Question:

$f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^3 - 1$

$f^{\leftarrow}(0)$?

a) $\{0\}$

b) 0

c) $\{1\}$ 41%

d) I

e) $\{-1, 1\}$ 41%

$$f^{\leftarrow}(0) = \{ x \in X : f(x) = 0 \}$$
$$= \{ x \in X : x^3 - 1 = 0 \}$$

$$x^3 - 1 = 0 \Rightarrow x^3 = 1$$
$$\Rightarrow x = 1$$

$f^{\leftarrow}(1) = \{x \in \mathbb{R} : x^3 - 1 = 1\}$

$x^3 - 1 = 1 \Rightarrow x^3 = 2$

$\Rightarrow x = 2^{1/3}$

$f^{\leftarrow}(y) = \{x \in \mathbb{R} : x^3 - 1 = y\}$

$x^3 - 1 = y \Rightarrow x^3 = y + 1$

$\Rightarrow x = (y+1)^{1/3}$

a) $f^{\leftarrow}(y) = (y+1)^{1/3}$

b) $f^{\leftarrow}(y) = \{(y+1)^{1/3}\}$ 68%

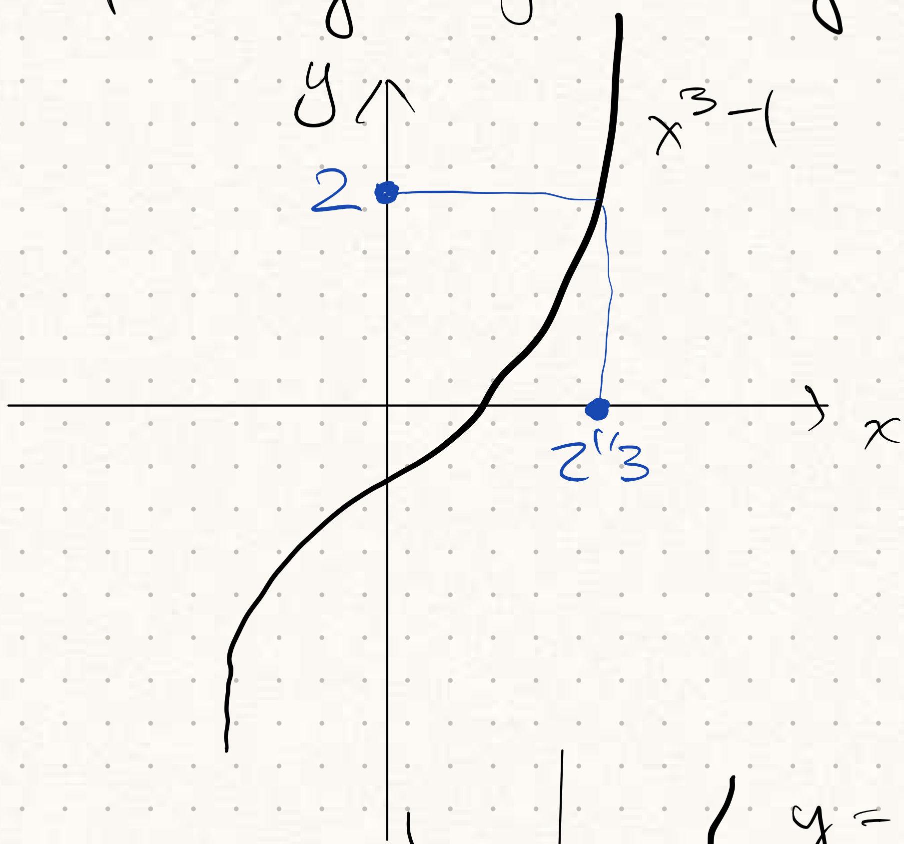
Answer is b)!

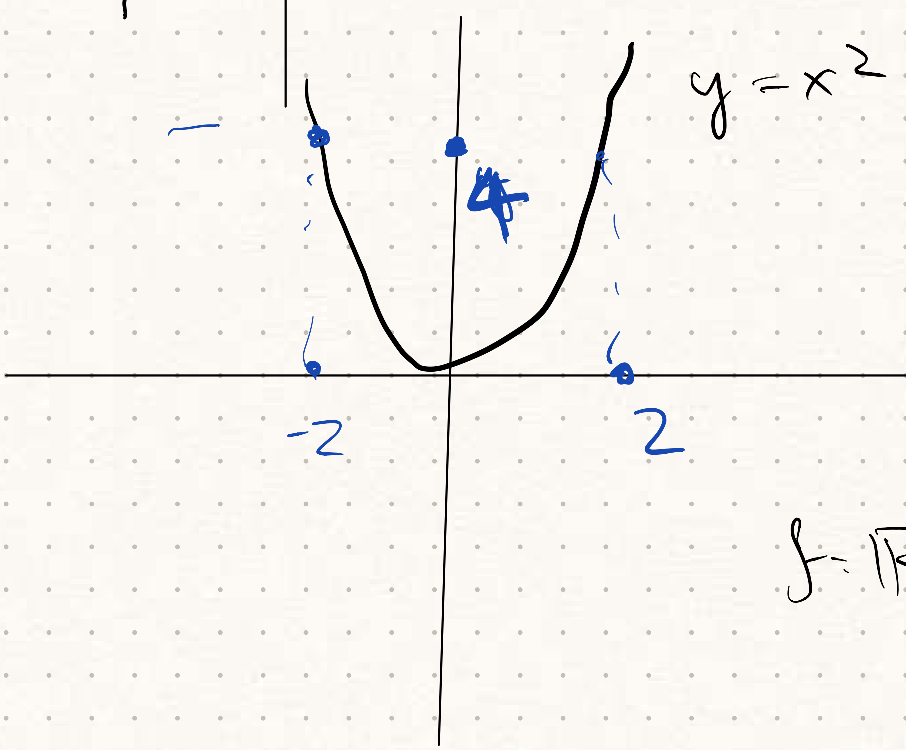
Is f surjective? 77% Yes.

For all y, $f^{\leftarrow}(y)$ has at least one elem.

Is it injective? Yes.

Graphically injectivity / surjectivity:





$f: \mathbb{R} \to \mathbb{R}$

"Horizontal line test!"

Different ways of saying injectivity and surjectivity.

Question

$f: X \to Y$ surjective $\iff$

a) $\forall y \in Y \ \forall x \in X \ f(x) = y$

66% b) $\forall y \in Y \ \exists x \in X \ f(x) = y$

26% c) $\exists y \in Y \ \forall x \in X \ f(x) = y$

d) $\exists y \in Y \ \exists x \in X \ f(x) = y$

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b) For all $y \in Y$
There exists $x \in X$
$f(x) = y$ ✓

c) There exists $x \in X$
For all $y \in Y$
$f(x) = y$ ✗

$f: X \to Y$ injective $\iff$

32% a) $x = y \Rightarrow f(x) = f(y)$

58% b) $f(x) = f(y) \Rightarrow x = y$

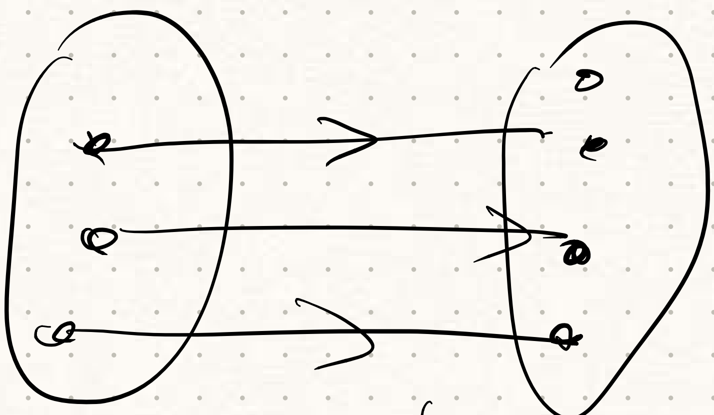
14% c) $x = y \Rightarrow f(x) \neq f(y)$

50% d) $x \neq y \Rightarrow f(x) \neq f(y)$

a) if $x = y$ then $f(x) = f(y)$ ✗

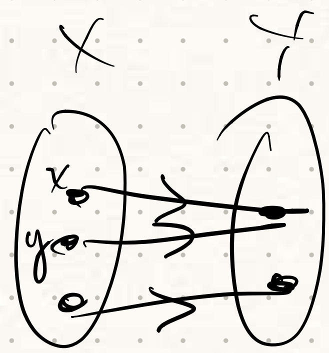
true for all functions anyway.

b) if $f(x) = f(y)$ then $x = y$.



Injective.

(not surjective)



not injective.

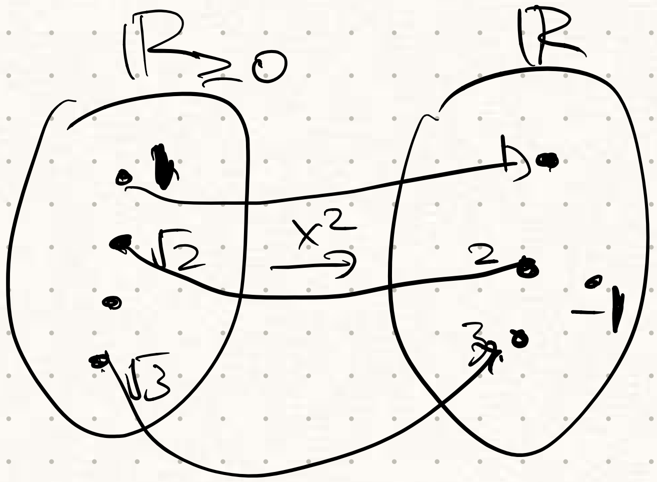
d) contrapositive.

Prove that for

$$f : \mathbb{R}_{\geq 0} \to \mathbb{R}$$

$$f(x) = x^2 \text{ is injective.}$$

Rough idea



injective.

Proof Want to show

$$f(x) = f(y) \implies x = y$$

Suppose $f(x) = f(y)$.

Then $x^2 = y^2$

So $x^2 - y^2 = 0$

So $(x+y)(x-y) = 0$

So $(x+y) = 0$ or $(x-y) = 0$

So $x = -y$ or $x = y$.

impossible, because $x \geq 0$ and $y \geq 0$

So $x = y$.

Excercise

$f: \mathbb{R} \to \mathbb{R}$

$f(x) = x + 3$

a) Prove that f is injective.

b) Prove that f is surjective

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a) Want to show

For $x \in \mathbb{R}$, $y \in \mathbb{R}$, $f(x) = f(y) \Rightarrow x = y$

Assume $f(x) = f(y)$.

So $x + 3 = y + 3$

So $x + 3 - 3 = y + 3 - 3$

So $x = y$.

b) Prove that f is surjective.

Want to show

$$\forall y \in Y \quad \exists x \in X \quad f(x) = y \qquad (1)$$

Suppose $y \in Y$.

Want to find x
for which $f(x) = y$.

Want to find x
for which $x + 3 = y$.

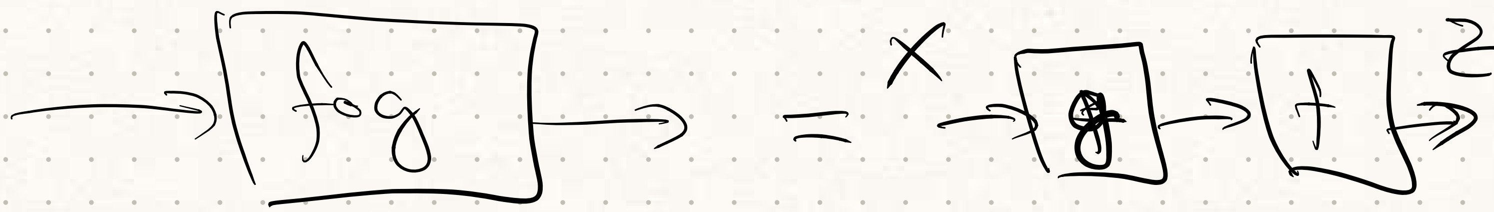
Take $x = y - 3$.

This proves (1).

Function composition

If $f : Y \to Z$ and $g : X \to Y$

$f \circ g$ is the function



$$(f \circ g)(x) = f(g(x))$$

What is the domain and range of $f \circ g$?

a) $f \circ g : X \to X$
b) $f \circ g : X \to Y$
c) $f \circ g : Y \to Y$
d) $f \circ g : X \to Z$ 88%.
e) $f \circ g : Z \to X$

Example

$\text{motherOf} : \text{Humans} \to \text{Humans}$

$\text{motherOf}(x) =$ the mother of x

$\text{fatherOf} : \text{Humans} \to \text{Humans}$

$\text{fatherOf}(x) =$ the father of x.

$\text{motherOf} \circ \text{fatherOf}(x)$

= a) The maternal grandmother of x

65% b) The paternal grandmother of x

c) The paternal grandfather of x

d) The maternal grandfather of x.

Answer is b)

$\text{motherOf} \circ \text{fatherOf}(x)$

$= \text{motherOf}($father of $x)$

= mother of father of x.

Example

$\text{motherOf} \circ \text{fatherOf} \circ \text{fatherOf} \circ \text{motherOf}(x)$

How would you say this in english?

<u>Inverses</u> Suppose

$f : X \to Y$

$g : Y \to X$

Then g is a left inverse for f if

$$g \circ f = \mathrm{Id}_X$$

<u>Identity function.</u>

For a set X,

Id_X is a special function

$\mathrm{Id}_X : X \to X, \quad \mathrm{id}_X(a) = a$

Roughly speaking:

"g is left inverse of f"

means

"g undoes what f does".

or

"Doing f, then doing g, is the same as doing nothing."

Example firstChildOf: Parents $\to$ Humans.

firstChildOf(x) = the first child of x.

Is firstChildOf a left inverse for motherOf?

Result: 50/50 yes no.

Answer: no.

Is

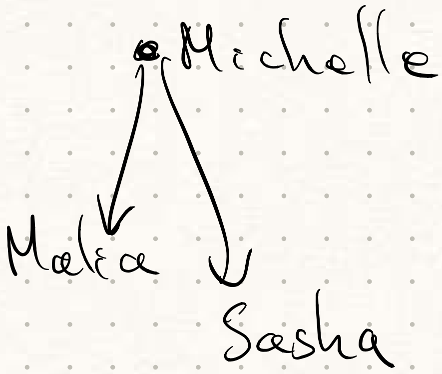
$$\text{firstChildOf} \circ \text{motherOf} = \text{Id}_{\text{Humans}}.$$

Is this true for all x.

$$\text{firstChildOf}(\text{motherOf}(x)) = \text{Id}_{\text{Humans}}(x)$$

$$\text{firstChildOf}(\text{motherOf}(x)) = x$$

No!



$$\text{firstChildOf}(\text{motherOf}(\text{sasha})) = \text{firstChild}(\text{Michelle})$$

$$= \text{Malia}.$$