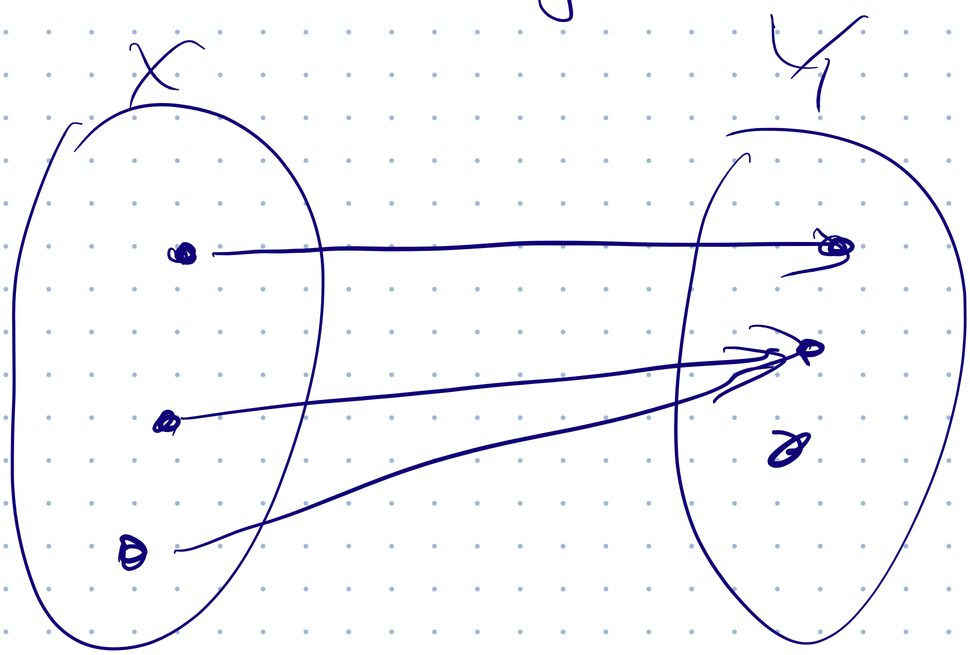
Lecture 14

Recall

* $f : X \to Y$

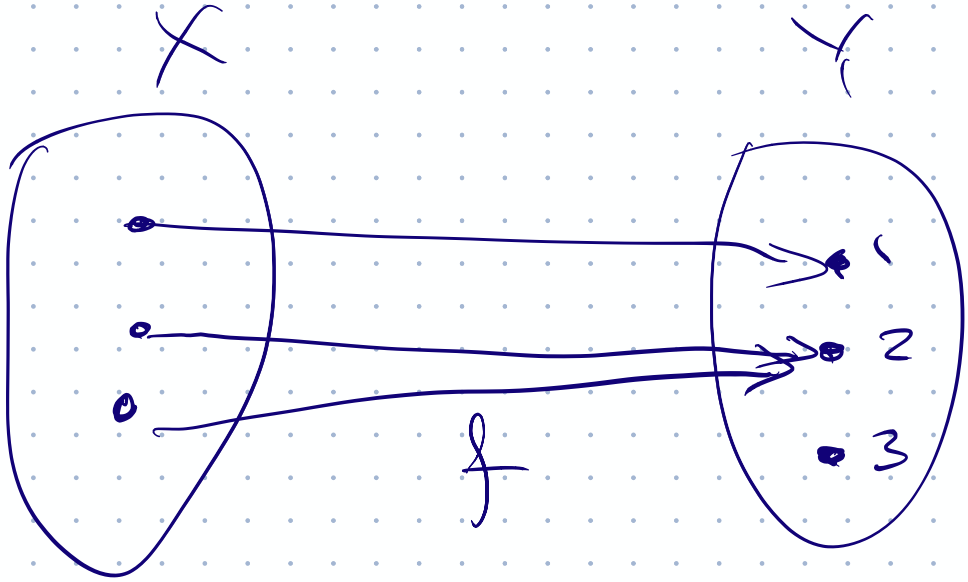
means f is a function, from domain X to range Y.



Definition image of f

$$\operatorname{im} f = \{ f(x) : x \in X \}$$

E.g.



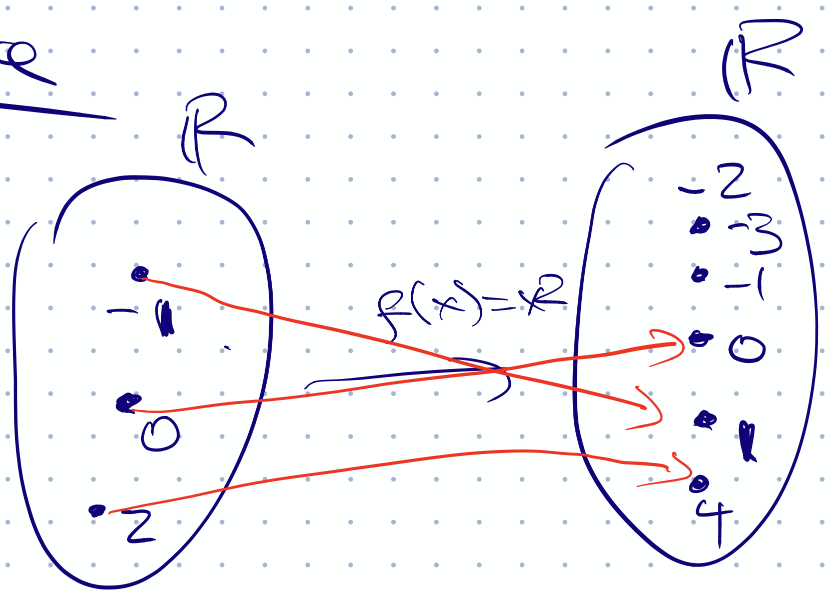
$\operatorname{Im} f = \{1, 2\}$

E.g. $f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$

$\operatorname{Im} f = \mathbb{R}$?

False



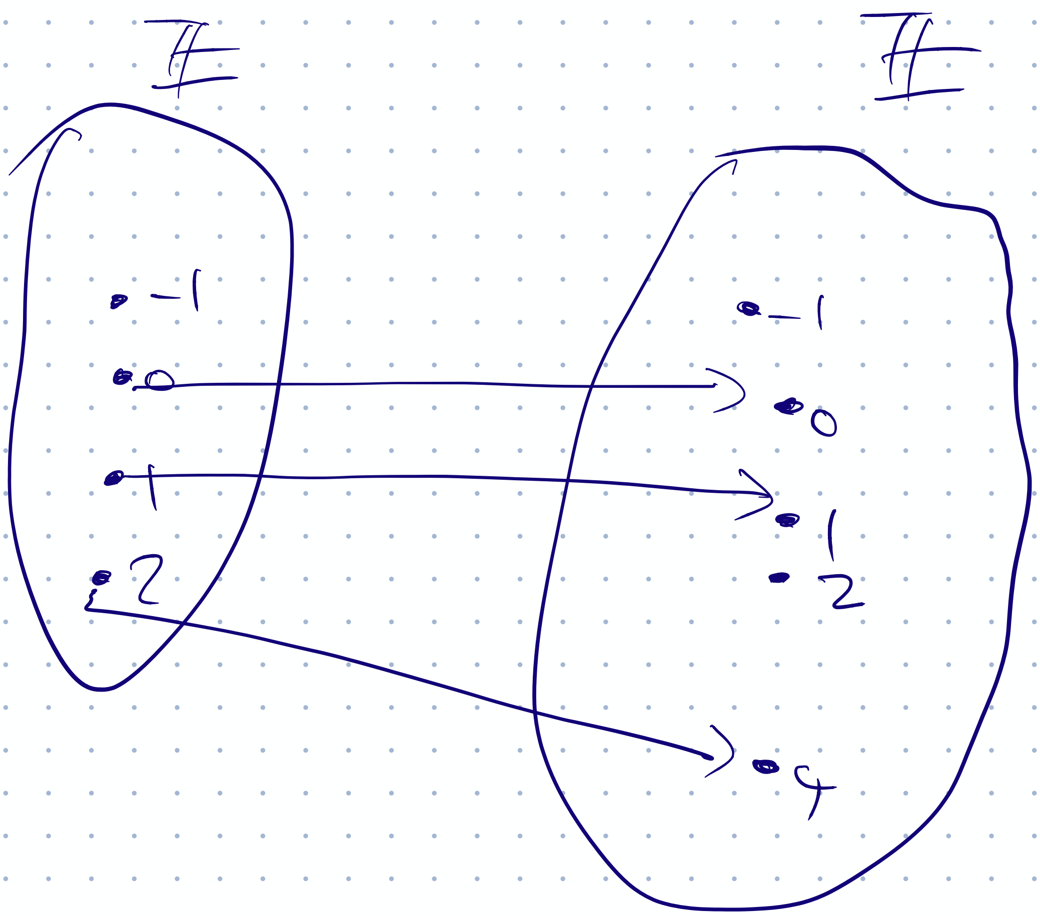
$\operatorname{Im} f =$
$\{x \in \mathbb{R} : x \geq 0\}$

Ex $\quad f: \mathbb{Z} \to \mathbb{Z}$

$$f(x) = x^2$$

$$\operatorname{Im} f = \{x \in \mathbb{Z} : x \geq 0\} \quad ?$$

False.



$$\operatorname{Im} f = \{0, 1, 4, 9, 16, \ldots\}.$$

$\operatorname{Im} f \neq \{x \in \mathbb{Z} : x \geq 0\}$ because

$2 \geq 0$, but $2 \notin \operatorname{Im} f$.

$\operatorname{Im} f =$

Is there a way to express "the set of perfect squares in set notation".

One answer:

* $\operatorname{Im} f$ where $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$.

Second answer:

* $\{ x^2 : x \in \mathbb{Z} \}$

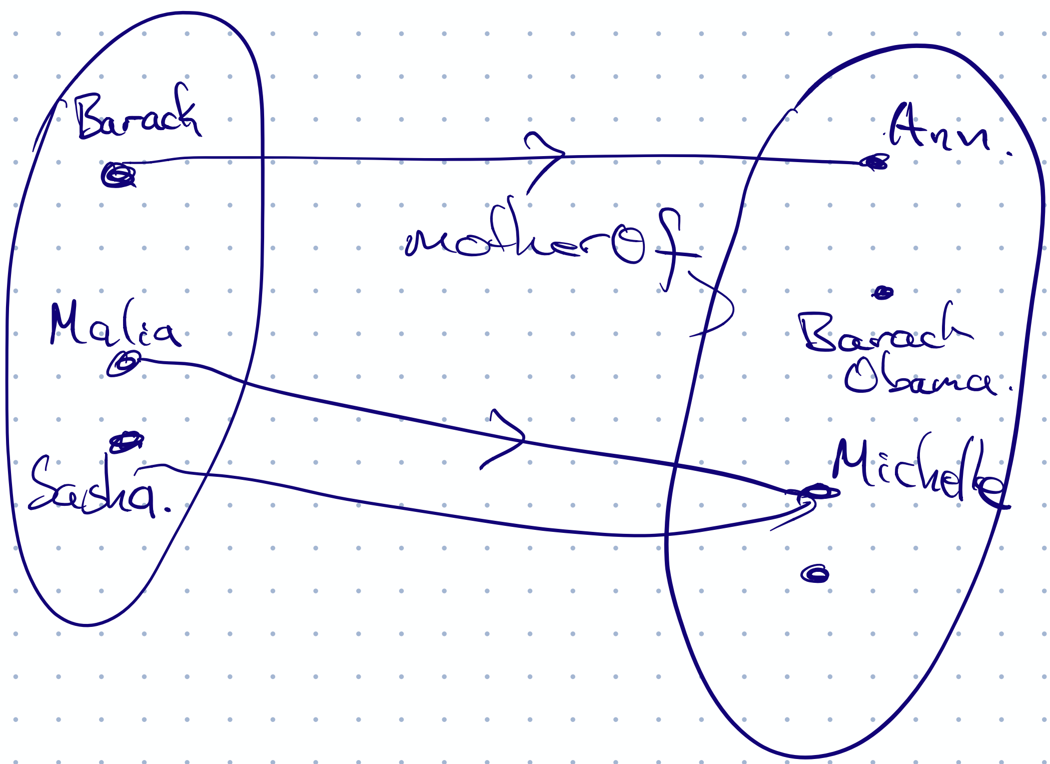
E.g. $\text{motherOf} : \text{Humans} \to \text{Humans}$

$\text{motherOf}(x) =$ the mother of x.

$\text{Im}(\text{motherOf}) = \,?$

is barack obama $\in \text{im}(\text{MotherOf})$.

No. Humans. Humans



Not asking for $\text{motherOf}(\text{barack})$.

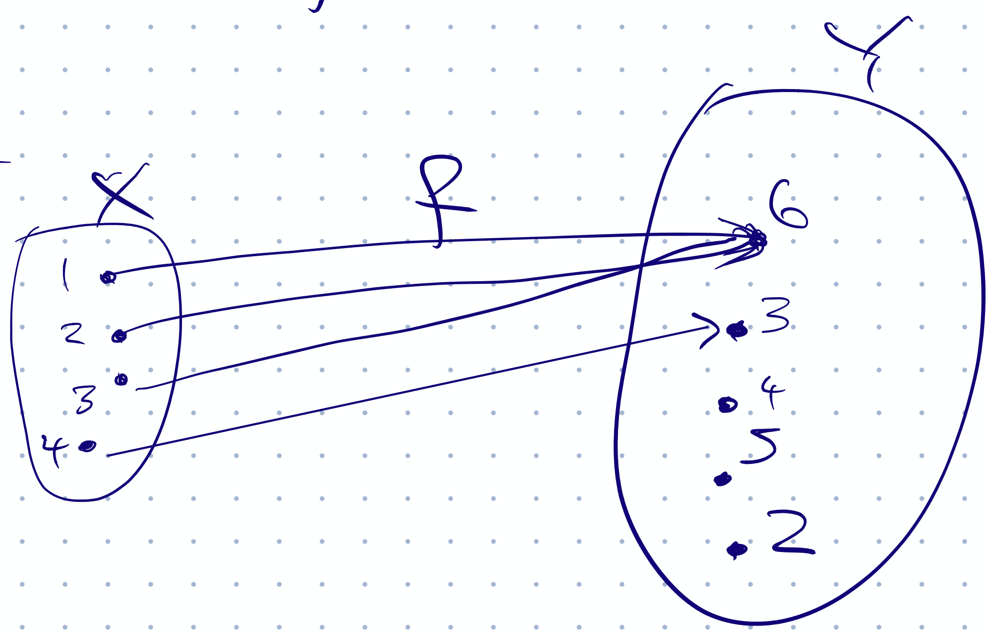
Asking if $\text{MotherOf}(x) = \text{barack}$.

Pre-images

For $f: X \to Y$, $y \in Y$

$$\underbrace{f^{\leftarrow}(y)}_{\text{pre-image of } y \text{ under } f} = \{x \in X : f(x) = y\}.$$

Picture:



$$f^{\leftarrow}(6) = \{x \in X : f(x) = 6\} = \{1,2,3\}.$$

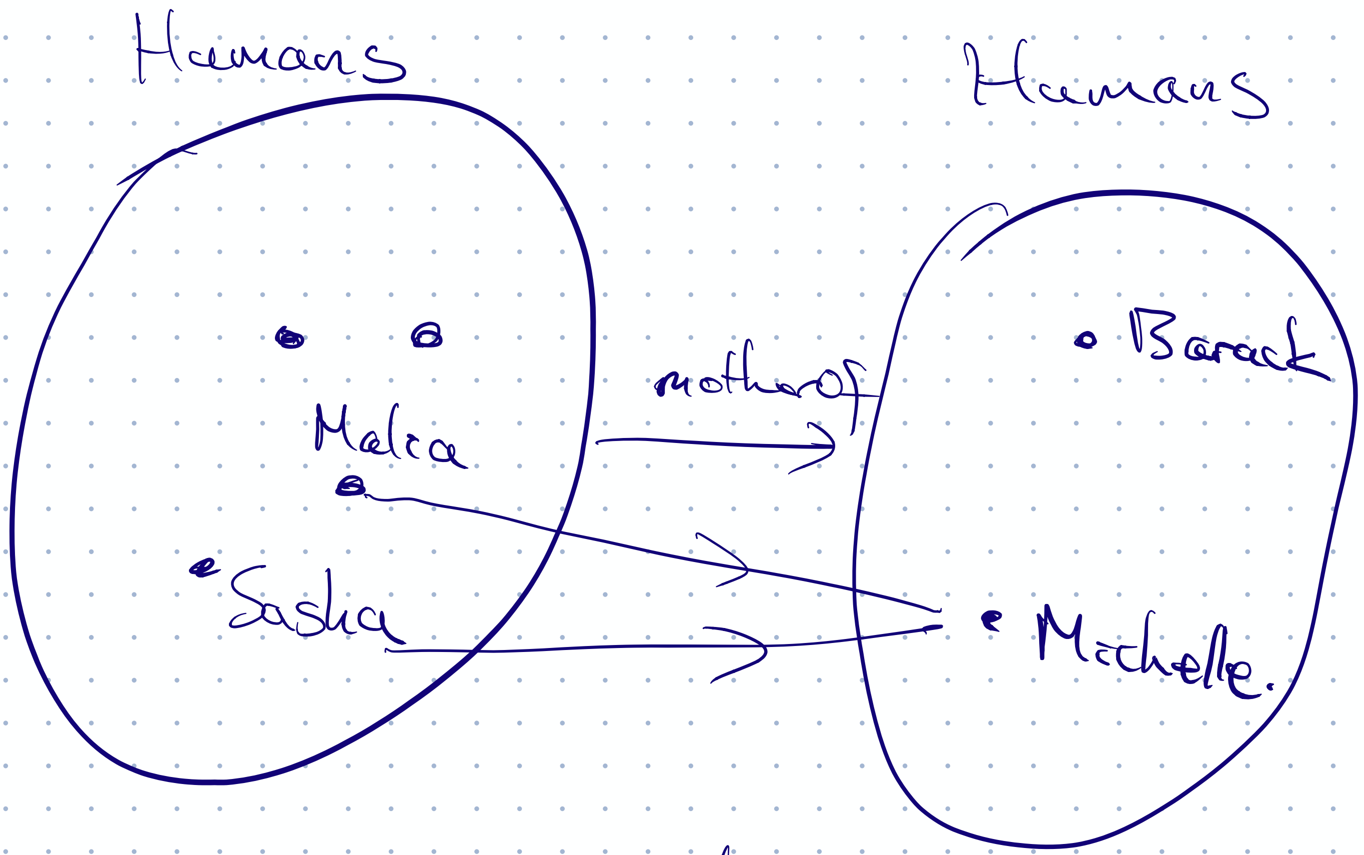
$$f^{\leftarrow}(3) = \{x \in X : f(x) = 3\} = \{4\}$$

$$f^{\leftarrow}(2) = \{x \in X : f(x) = 2\} = \emptyset.$$

Example

$$\text{motherOf}^{\leftarrow}(\text{Barack Obama}) = \emptyset$$

$$\text{motherof}^{\leftarrow}(\text{Michelle Obama}) = \{\text{Sasha}, \text{Malia}\}.$$



$$\text{motherOf}^{\leftarrow}(x) = \text{children of } x.$$

E.g.

$f(x) = x^2 \quad f: \mathbb{R} \to \mathbb{R}$

$$f^{\leftarrow}(9) = \{x \in \mathbb{R} : f(x) = 9\}$$
$$= \{-3, 3\}.$$

$f(x) = x^2, \quad f: \mathbb{R}_{\geq 0} \to \mathbb{R}$

$$f^{\leftarrow}(9) = \{x \in \mathbb{R}_{\geq 0} : f(x) = 9\}$$
$$= \{3\}.$$

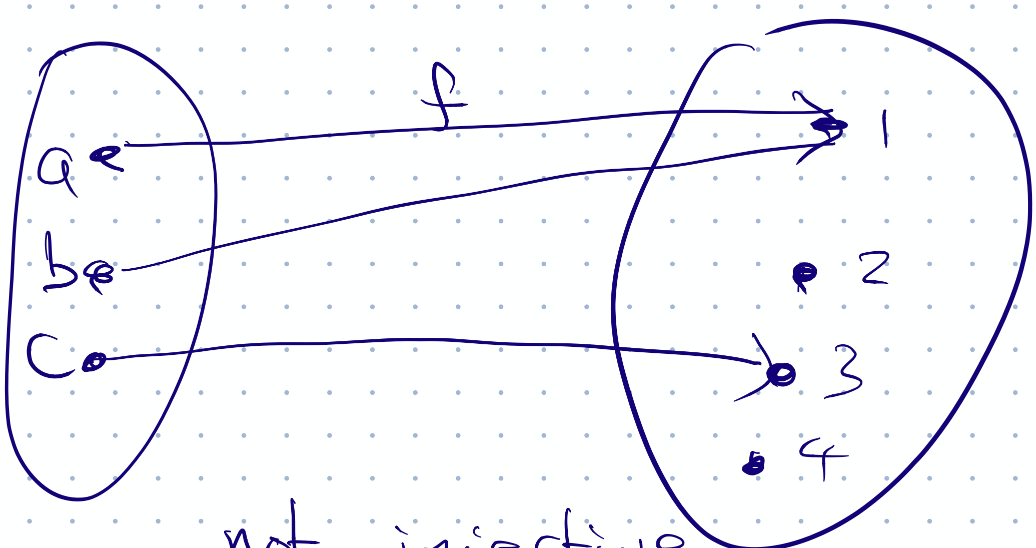
Injective functions

$f : X \to Y$ is *injective* if

For all $y \in Y$

$f^{-1}(y)$ has at most one point

Picture



not injective.

$f^{-1}(1) = \{a, b\}$, more than 1

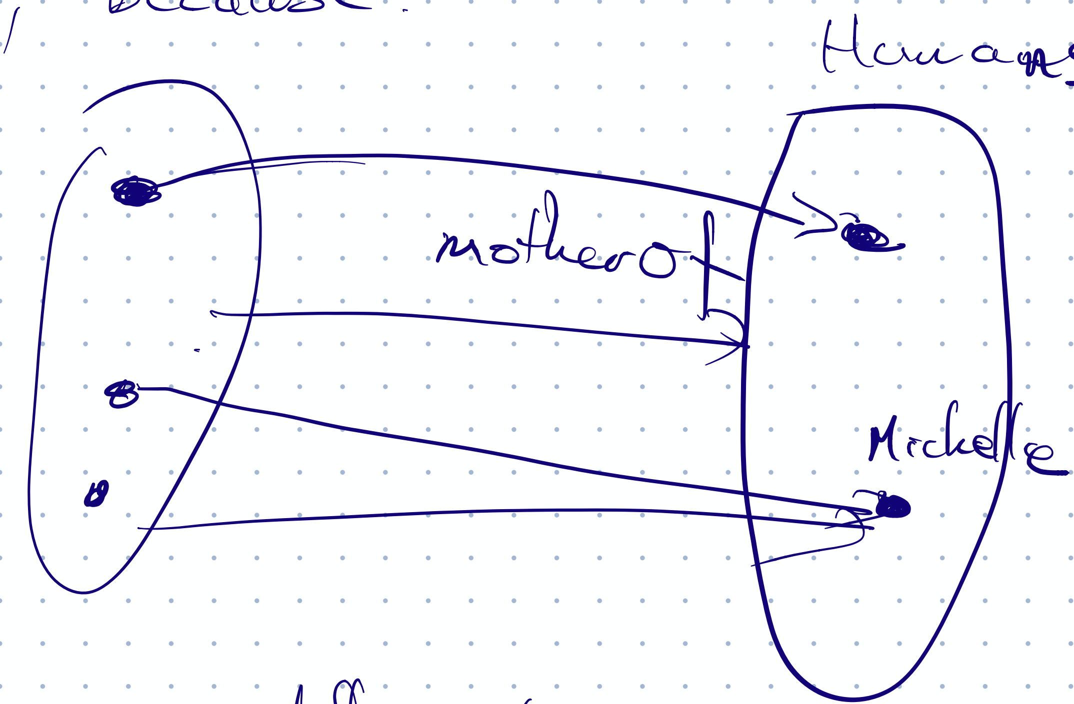
Roughly speaking: injective means there is only one (or zero) things pointing to every element in the range.

Example

$$\text{motherOf}: \text{Humans} \to \text{Humans}$$

is motherOf injective?

No, because.



or, different ~~people~~ Humans can have the same mother.

$f(x) = x^2$

$f: \mathbb{R} \to \mathbb{R}$ injective?

No. Proof: $f^{\leftarrow}(1) = \{-1, 1\}$.

$f : \mathbb{R}_{\geq 0} \to \mathbb{R}$

$f(x) = x^2$ injective.

Yes.

$f^{\leftarrow}(1) = \{1\}$

$f^{\leftarrow}(2) = \{\sqrt{2}\}$

$f^{\leftarrow}(-1) = \emptyset$

Reason:

Every real number either has no square root.

Or has one positive sqrt.

Surjective functions

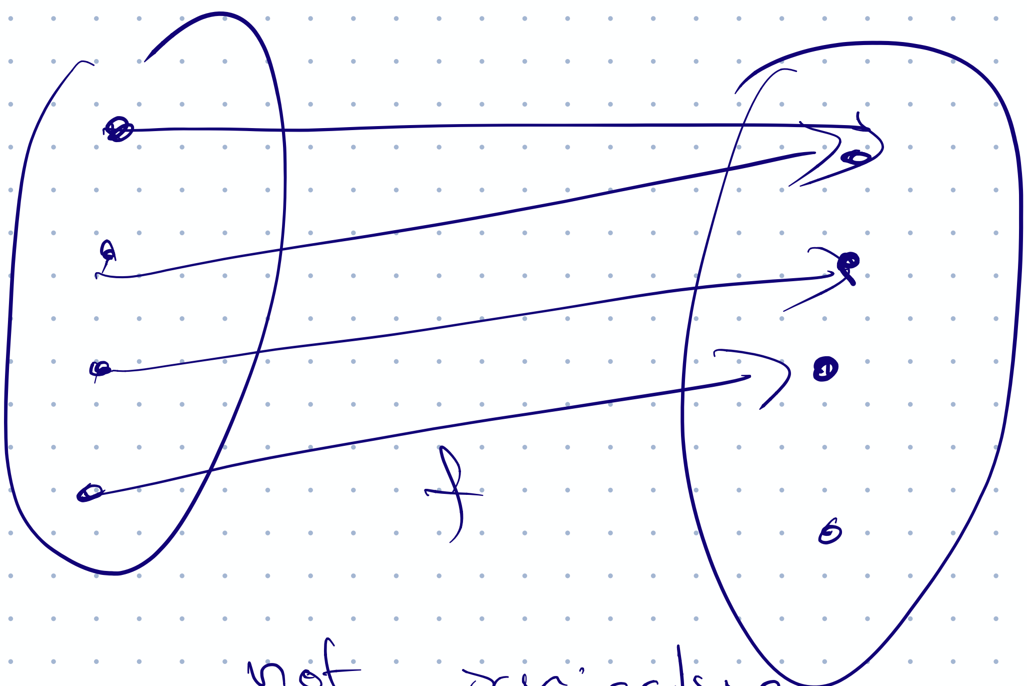
$f: X \to Y$ is surjective

if

For all $y \in Y$, $f^{\leftarrow}(y)$

has at least one element.

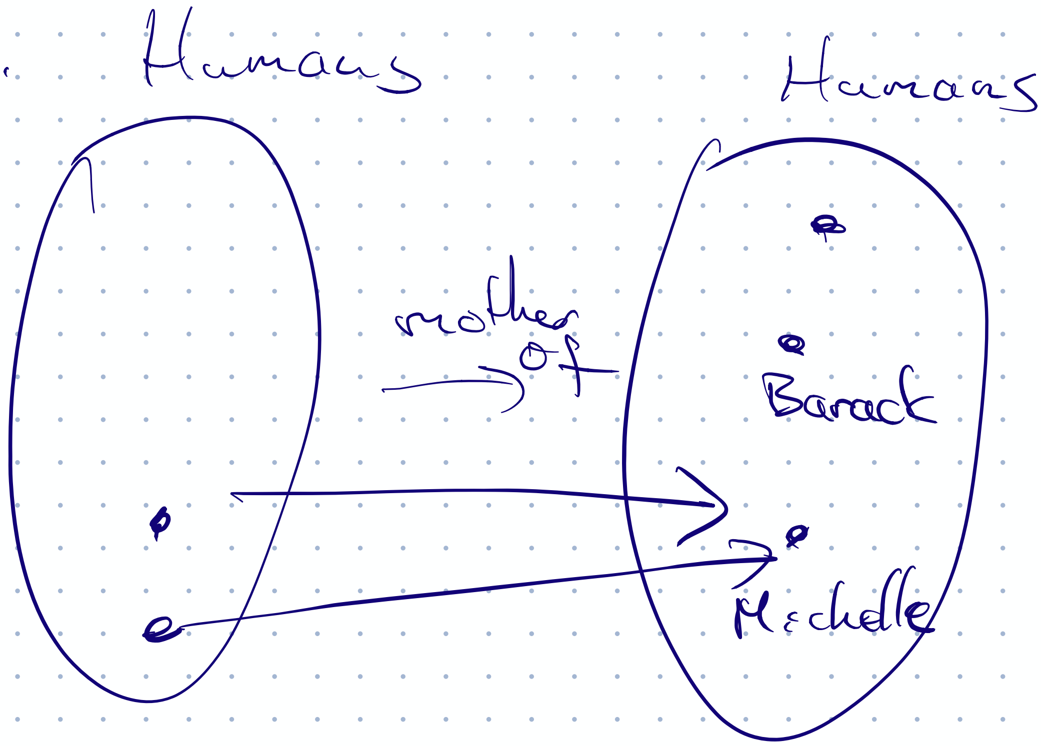
Picture:



not surjective.

is motherOf surjective?

No.



$\text{motherOf}^{\leftarrow}(\text{Barack}) = \emptyset$,

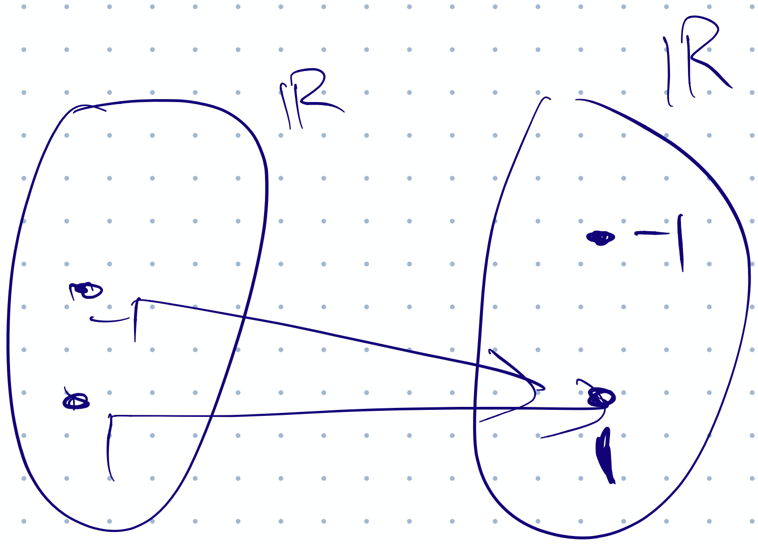
so motherOf is not surjective.

Is $f(x) = x^2$

$f: \mathbb{R} \to \mathbb{R}$

surjective?

no.



$f^{\leftarrow}(-1) = \emptyset$ so f not surj.

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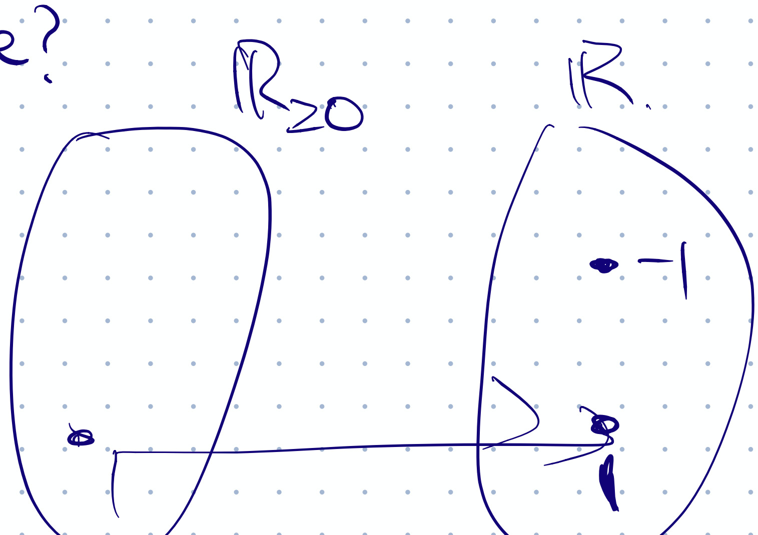
is $f(x) = x^2$

$f: \mathbb{R}_{\geq 0} \to \mathbb{R}$

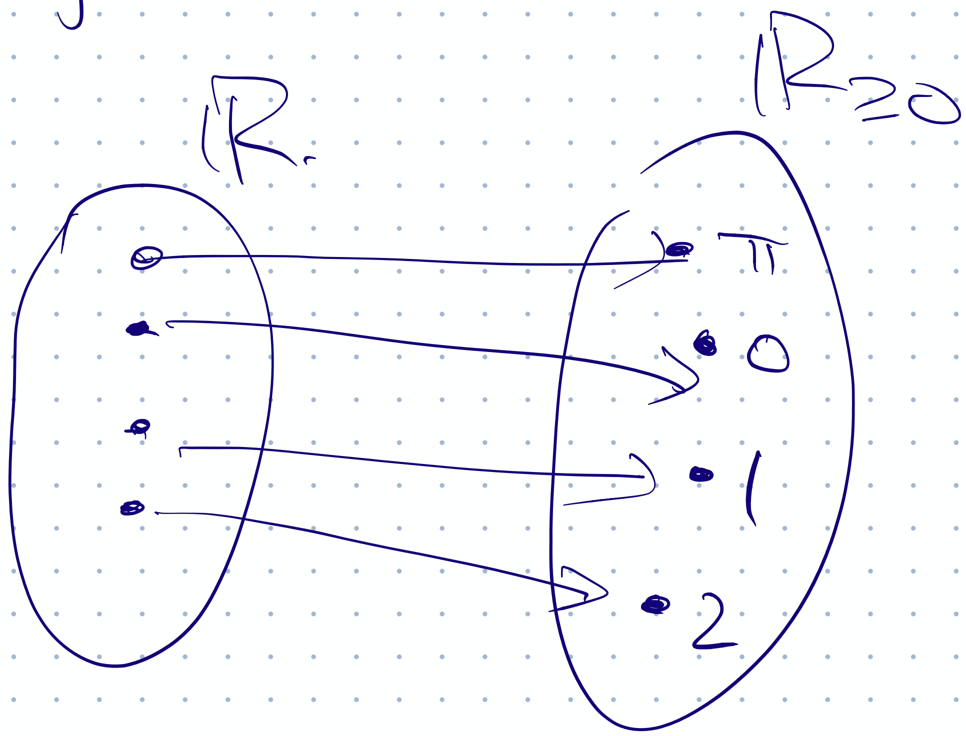
surjective?

no



$f : \mathbb{R} \to \mathbb{R}_{\geq 0}$

f surjective.



Yes surjective.

Image

$$f : X \to Y$$
$$A \subset X$$

$$\vec{f}(A) := \{ f(x) : x \in A \}$$

set

$$\vec{f} : P(X) \to P(Y).$$

A