MAT 402

Homework VI
Due May 9th. 2019. Show all your work

(1) Let $U := \{z \in \mathbb{C} : |z| < 1\}$, and let $\mathcal{F} = (f_n)_{n=1}^{\infty}$ be the family of polynomials on U defined by

$$
\begin{aligned}
f_n : U &\mapsto \mathbb{C} \\
f_n(z) &= z^2 + z + 4 + n
\end{aligned}
$$

Use Montel's theorem to prove that $\mathcal{F}$ is normal on U.

(2) Let $D \subseteq \mathbb{C}$ be a domain, and let $\mathcal{F}$ be the family of all injective holomorphic functions on D that omit the value of 0.

Use the Zalcman lemma to prove that $\mathcal{F}$ is normal.

(3) Suppose that $D \subseteq D' \subseteq \mathbb{C}$ are domains, and that D' is convex. Suppose also that $\varphi : D' \to D$ such that $|\varphi'(z)| \leq \lambda < 1$, for $z \in D'$. Prove that

$$
|\varphi(z) - \varphi(w)| \leq \lambda |z - w| \, ,
$$

for $z, w \in D'$. Deduce by the Banach fixed point theorem, that φ has a (unique) fixed point $p \in D'$.

Hint. Express $|\varphi(z) - \varphi(w)|$ as an integral.