A HEURISTIC PRINCIPLE IN COMPLEX FUNCTION THEORY

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1. Introduction. A well-known heuristic principle in the theory of functions asserts that "a family of holomorphic (meromorphic) functions which have a property P in common in a domain D is (apt to be) a normal family in D if P cannot be possessed by non-constant entire (meromorphic) functions in the finite plane" [**4**, p. 250]. In his recent retiring presidential address to the Association for Symbolic Logic [**7**] (required reading for anyone whose interests extend across professional boundaries to mathematics as an intellectual discipline), the late Professor Abraham Robinson cited the explication of this principle as one of twelve problems worthy of the attention of logicians (and, by extension, of mathematicians in general).

This paper is devoted to such an explication. To be precise, we prove a simple theorem which makes the principle rigorous and from which the standard applications of the principle follow quite routinely. Some of these applications are noted at the end, together with a slightly novel approach to the proof of Picard's big theorem.

Major credit for the mathematical content of this paper belongs to Christian Pommerenke. He proved a result similar to the main lemma for *normal functions* [**6**, Theorem 1]. It turns out that the same proof works in the (more general) context of normal families and even simplifies a little. Thus, this paper is perhaps viewed most properly as a public relations effort for Pommerenke's theorem.

2. Normal families. A family $\mathscr{F}$ of functions holomorphic on a domain $D \subseteq \mathbf{C}$ is said to be *normal* on D if each sequence of functions in $\mathscr{F}$ contains a subsequence which either converges (to some function not necessarily in $\mathscr{F}$) uniformly on every compact subset of D or tends uniformly to infinity on each compact subset of D. For meromorphic functions, it is advantageous to introduce the familiar *spherical*, or *chordal*, *metric*

$$\chi(z, z') = \frac{|z - z'|}{(1+|z|^2)^{1/2}(1+|z'|^2)^{1/2}} \qquad \chi(z, \infty) = \frac{1}{(1+|z|^2)^{1/2}}$$

on the Riemann sphere. A family $\mathscr{F}$ of meromorphic functions on D is then said to be normal if each sequence of functions in $\mathscr{F}$ has a subsequence which converges uniformly on compacta *with respect to the spherical metric.*

The spherical distance and the ordinary Euclidean absolute value are boundedly equivalent on compact subsets of the plane. It follows that if $\{f_n\}$ converges spherically uniformly on a set S to the limit function $f \not\equiv \infty$, then the f_n actually converge to f uniformly on any compact subset of S disjoint from the poles of f. This observation (and a little thought) shows that for analytic (i.e., holomorphic) functions the two definitions of normality agree. See [**1**, pp. 210–219] for an illuminating discussion of these matters.

We shall need the following characterization of normal families, due to F. Marty.

THEOREM. *A family $\mathscr{F}$ of functions analytic or meromorphic on D is normal if and only if the functions*

$$f^{\#}(z) = \frac{|f'(z)|}{1+|f(z)|^2} \qquad f \in \mathscr{F}$$

are uniformly bounded on each compact subset of D.

Here $f^{\#}$ is the *spherical derivative* of f, sometimes denoted by $\rho(f)$; the present notation (used in [**6**]) is better adapted for displaying the argument of the function explicitly. At the poles of f, $f^{\#}$ is defined by continuity; equivalently, one may use the relation $f^{\#} = (1/f)^{\#}$, obviously valid at regular points of f and $1/f$, to define $f^{\#}$ throughout all of D. The spherical derivative has an appealing geometric interpretation: one has

$$f^{\#}(z)=\lim_{h\to 0}\chi(f(z+h),f(z))/|h|,$$

and if γ is an are in D, then $\int_\gamma f^{\#}(z)|dz|$ measures the length of $f(\gamma)$ on the Riemann sphere. Observe that $f^{\#}(z)=0$ only if $f'(z)=0$ or $f(z)=\infty$.

Various proofs of Marty's theorem exist in the literature. Marty's original, rather geometric, argument can be found in Ahlfors [**1**, pp. 218–219]; a purely analytic proof is in [**3**, pp. 158–160].

3. The main lemma. Before formulating our principle, it will be convenient to give an alternative characterization of normality, which makes explicit the relation with entire or meromorphic functions on the plane. The following lemma is perhaps best understood in the context of non-standard analysis (for which connection see [**7**, pp. 509–510]); crudely put, it says that in the absence of normality a certain kind of infinitesimal convergence must take place.

LEMMA. *A family $\mathcal{F}$ of functions meromorphic* [*analytic*] *on the unit disc* Δ *is not normal if and only if there exist*

(a) *a number* $0<r<1$

(b) *points* z_n, $|z_n|<r$

(c) *functions* $f_n\in\mathcal{F}$

(d) *numbers* $\rho_n\to 0+$

such that

$$f_n(z_n+\rho_n\zeta)\to g(\zeta) \tag{1}$$

spherically uniformly [*uniformly*] *on compact subsets of* **C**, *where g is a nonconstant meromorphic* [*entire*] *function on* **C**.

Proof. Suppose $\mathcal{F}$ is not normal on Δ. Then by Marty's theorem there exists a number r^*, $0<r^*<1$, points z_n^* in $\{z:|z|\leqq r^*\}$, and functions $f_n\in\mathcal{F}$ such that $f_n^{\#}(z_n^*)\to\infty$. Fix a number r, $r^*<r<1$, and let

$$M_n=\max_{|z|\leqq r}\left(1-\frac{|z|^2}{r^2}\right)f_n^{\#}(z)=\left(1-\frac{|z_n|^2}{r^2}\right)f_n^{\#}(z_n). \tag{2}$$

The maximum exists since $f_n^{\#}$ is continuous for $|z|\leqq r$, and it is clear that $M_n\to\infty$. Setting

$$\rho_n=\frac{1}{M_n}\left(1-\frac{|z_n|^2}{r^2}\right)=\frac{1}{f^{\#}(z_n)}, \tag{3}$$

we obtain

$$\frac{\rho_n}{r-|z_n|}=\frac{r+|z_n|}{r^2M_n}\leqq\frac{2}{rM_n}\to 0. \tag{4}$$

Thus, the functions

$$g_n(\zeta)=f_n(z_n+\rho_n\zeta)$$

are defined for $|\zeta|<R_n$, where $R_n=(r-|z_n|)/\rho_n\to\infty$ as $n\to\infty$. It follows from (3) that

$$g_n^{\#}(0)=\rho_nf_n^{\#}(z_n)=1.$$

For $|\zeta|\leqq R<R_n$, $|z_n+\rho_n\zeta|<r$ so that by (2) and (3)

$$g_n^{\#}(\zeta)=\rho_nf_n^{\#}(z_n+\rho_n\zeta)\leqq\frac{\rho_nM_n}{1-\frac{|z_n+\rho_n\zeta|^2}{r^2}}\leqq\frac{r+|z_n|}{r+|z_n|+\rho_nR}\cdot\frac{r-|z_n|}{r-|z_n|-\rho_nR}.$$

The first factor on the right is bounded by 1, while (for fixed R) the second tends to 1 as $n \to \infty$ by (4). Thus, by Marty's theorem, $\{g_n\}$ is a normal family; taking a subsequence, we may assume that the g_n converge uniformly (in the spherical metric) on compact subsets of $\mathbf{C}$ to a meromorphic function g. Finally, g is nonconstant since $g^{\#}(0) = \lim g_n^{\#}(0) = 1 \neq 0$. It is now clear that if $\mathcal{F}$ consists of analytic functions the limit function will be entire.

For the converse, suppose that $\mathcal{F}$ is normal on Δ. By Marty's theorem, there exists $M > 0$ such that

$$\max_{|z| \leqq (1+r)/2} f^{\#}(z) \leqq M$$

for all $f \in \mathcal{F}$. Suppose (1) holds and fix $\zeta \in \mathbf{C}$. For large n, $|z_n + \rho_n \zeta| \leqq (1+r)/2$, so that $\rho_n f_n^{\#}(z_n + \rho_n \zeta) \leqq \rho_n M$. Thus, for all $\zeta \in \mathbf{C}$,

$$g^{\#}(\zeta) = \lim \rho_n f_n^{\#}(z_n + \rho_n \zeta) = 0.$$

It follows that g is constant (possibly infinity).

As noted earlier, the proof of the lemma is virtually identical to Pommerenke's proof of a slightly different result. The clever trick of using a cut-off function $(1 - (|z|^2/r^2))$ can be traced back at least to Landau's proof of Bloch's theorem [**5**, p. 99].

4. A matter of principle. To formulate the heuristic principle precisely, it will be convenient to follow Robinson's idea of displaying the domain of definition of a function explicitly together with the function. Thus, we write $\langle f, D \rangle$ to denote the function f defined on the domain $D \subset \mathbf{C}$, and we distinguish between the functions $\langle f, D \rangle$ and $\langle f, D' \rangle$ if $D \neq D'$

The principle may then be stated as follows.

THEOREM. *Let P be a property (i.e., a set) of meromorphic [holomorphic] functions which satisfies the following conditions.*

(i) *If $\langle f, D \rangle \in P$ and $D' \subset D$ then $\langle f, D' \rangle \in P$.*

(ii) *If $\langle f, D \rangle \in P$ and $\phi(z) = az + b$, then $\langle f \circ \phi, \phi^{-1}(D) \rangle \in P$.*

(iii) *Let $\langle f_n, D_n \rangle \in P$, where $D_1 \subset D_2 \subset D_3 \subset \cdots$ and $D = \cup D_n$. If $f_n \to f$ spherically uniformly on compact subsets of D, then $\langle f, D \rangle \in P$.*

Suppose $\langle f, \mathbf{C} \rangle \in P$ only if f is constant. Then for any domain D the family of functions satisfying $\langle f, D \rangle \in P$ is normal on D.

The present formulation differs from Robinson's (tentative) version [**7**, p. 509] in that it applies to meromorphic as well as analytic functions. More importantly, we only require invariance with respect to linear (as opposed to general conformal) maps.

Condition (i) is only a convenience; it can be avoided by a suitable reformulation of (iii). Conditions (ii) and (iii), on the other hand, are quite essential, as the following examples show. (Actually, (ii) need hold only for $0 < a \leqq 1$, and it would be enough to require (iii) only for the case where the D_n are discs centered at the origin and $D = \mathbf{C}$.)

EXAMPLE 1. Let $\langle f, D \rangle \in P$ if and only if $D \subset \Delta = \{z; |z| < 1\}$. Then (i) and (iii) are satisfied, while (ii) does not hold. Since $\langle f, \mathbf{C} \rangle \in$ P is never satisfied, P contains no entire or globally meromorphic functions at all. But the family of all analytic (or meromorphic) functions on Δ is clearly not normal (consider the sequence $\{f_n\}$, where $f_n(z) = nz$).

EXAMPLE 2. Let $\langle f, D \rangle \in P$ if and only if $D \neq \mathbf{C}$ ($D \subset \mathbf{C}$). For analytic functions, P may be phrased informally as "f is not entire." Obviously (i) and (ii) hold, but (iii) fails. Again P contains no entire functions; yet if D is any proper subdomain of $\mathbf{C}$, $\langle f, D \rangle \in P$ for all f analytic on D, and this family is not normal.

EXAMPLE 3. This is a more "natural" version of the preceding phenomenon. Let $\langle f, D\rangle \in P$ if and only if f is bounded on D, i.e., there exists a constant $M = M(f, D)$ such that $\sup_D |f(z)| \leqq M$. Conditions (i) and (ii) hold while (iii) fails. If $\langle f, \mathbf{C}\rangle \in P$, f must be constant by Liouville's theorem. But the family of all bounded analytic functions on a disc (for instance) is not normal.

A slight modification of the condition of Example 3 yields a positive result.

EXAMPLE 4. Fix $M > 0$ and let $\langle f, D\rangle \in P$ if and only if $\sup_D |f(z)| \leqq M$. Then (i)–(iii) are clearly satisfied and P contains no nonconstant entire functions. The theorem applies, and we recapture a classical sufficient condition for normality.

The proof of the theorem is hardly more than a restatement of the lemma of the preceding section, to which we refer the reader for the notation used below. Indeed, let $\mathcal{F}$ be the family of all functions on the domain which have property P. If $\mathcal{F}$ is not normal on D, Marty's condition (or the usual compactness argument) shows that it already fails to be normal on some subdisc, which (by (ii)) we may assume to be Δ. Let $R_n = (r - |z_n|)/\rho_n$; since $R_n \to \infty$, we may suppose (by taking a subsequence, if necessary) that the R_n form an increasing sequence. Set $g_n(\zeta) = f_n(z_n + \rho_n\zeta)$, $D_n = \{\zeta : |\zeta| < R_n\}$. The functions $\langle g_n, D_n\rangle$ satisfy P by (i) and (ii), so by (iii) $\langle g, \mathbf{C}\rangle$ does also. Since P contains no nonconstant functions defined on $\mathbf{C}$, this yields a contradiction. Thus, $\mathcal{F}$ must be normal on D.

5. An application. Perhaps the most celebrated criterion for normality is the following theorem, due to Paul Montel.

MONTEL'S THEOREM. *Let $\mathcal{F}$ be a family of functions meromorphic on the domain D. If there exist three points w_1, w_2, w_3 on the Riemann sphere such that $w_i \notin f(D)$ $(i = 1,2,3)$ for each $f \in \mathcal{F}$, then $\mathcal{F}$ is a normal family.*

Thus, Montel's theorem asserts that if each function in $\mathcal{F}$ omits the *same* three values then $\mathcal{F}$ is normal. (For families of *analytic* functions the value ∞ is always omitted, so one need require only that two finite values be omitted.) The usual proof makes use of Jacobi's elliptic modular function and is thus "nonelementary." Our principle, together with Picard's little theorem, gives an elementary proof. (Quite a different proof, also of elementary character, is in [**8**, pp. 347–350].)

Indeed, it is enough to take for P the property "either f is constant or it omits the values w_1, w_2, and w_3 on D." Conditions (i) and (ii) are at once seen to hold, while (iii) is a consequence of Hurwitz's theorem [**1**, p. 176] (or the argument principle). That any meromorphic function on $\mathbf{C}$ which satisfies P must be constant is, of course, Picard's little theorem.

Montel's theorem can be generalized in various directions. One extension, less well-known than it deserves, is the following.

EXTENDED MONTEL THEOREM. [**2**, vol. 2, p. 202] *Let $\mathcal{F}$ be a family of functions meromorphic on the domain D. Suppose that each $f \in \mathcal{F}$ omits three distinct values (which may depend on f) a, b, c on the sphere, the product of whose chordal distances $\chi(a,b)\chi(b,c)\chi(a,c)$ is bounded away from* 0 *independently of f. Then $\mathcal{F}$ is a normal family.*

For the proof, let ε be a positive lower bound for the product of the distances and take P to be the property "f omits three values a, b, c such that $\chi(a,b)\chi(b,c)\chi(a,c) \geqq \varepsilon$." By Picard's theorem, no nonconstant meromorphic function can have P. Thus, since (i) and (ii) are trivially satisfied, it remains only to show that P is preserved under uniform convergence with respect to the spherical metric.

Suppose then that $f_n \to f$ spherically uniformly on compact subsets of D and that f_n omits a_n, b_n, c_n with $\chi(a_n,b_n)\chi(b_n,c_n)\chi(a_n,c_n) \geqq \varepsilon$. We may assume f is nonconstant, for otherwise it trivially satisfies P. Since the sphere is compact, we can find points a, b, c and a subsequence (again denoted $\{f_n\}$) such that $\chi(a_n, a) \to 0$, $\chi(b_n, b) \to 0$, $\chi(c_n, c) \to 0$. By continuity, $\chi(a,b)\chi(b,c)$

$\chi(a,c) \geqq \varepsilon$, so we only need to prove that f never takes on the values a, b, c. Indeed, suppose $f(z_0) = a$, where $a \neq \infty$. Choose $r > 0$ such that $K = \{z : |z - z_0| \leqq r\} \subset D$ and f is analytic on K. Since f is bounded on K (and $a \neq \infty$), $f_n(z) - a_n$ converges *uniformly* on K to $f(z) - a$. The latter function is nonconstant and vanishes for $z = z_0$, so by Hurwitz's theorem $f_n(z) - a_n$ must (for large n) vanish on K. This is a contradiction. If $a = \infty$, we consider the functions $1/f$, $1/f_n$ and argue as before, using the invariance property $\chi(z, z') = \chi(1/z, 1/z')$.

6. Pedagogics. Montel's theorem is the central device in one of the standard proofs of the Big Picard Theorem: *in the neighborhood of an (isolated) essential singularity, a meromorphic function takes on every value in the Riemann sphere infinitely often with at most two exceptions.* For analytic functions, even more is true, as was proved by Gaston Julia.

JULIA'S THEOREM. *Let $f(z)$ be analytic in $D = \{z : 0 < |z| < 1\}$ with an essential singularity at* 0. *Then there exists a point $z_0 \in D$ such that, for each $\varepsilon > 0$, $f(z)$ assumes every complex value, with at most one exception, infinitely often on the union of the homothetic discs*

$$D_n = \{z : |z - z_0/2^n| < \varepsilon/2^n\}.$$

Again, the main tool in the proof is Montel's theorem. (The reader should be warned that the proofs given in [**4**, p. 259] and the first two editions of [**8**] are incomplete; a correct proof is in [**8**, pp. 351–352]).

We see no particular merit in avoiding the use of the modular function, which is at any rate required to obtain the precise values of the constants appearing in the theorems of Schottky and Landau [**2**, vol. 2, pp. 195–201]. On the other hand, it is perhaps of methodological interest that the theorems of both Picard and Julia can be given a purely elementary proof. One such development may be found in the important text of Saks and Zygmund [**8**, pp. 341–353].

An alternate program for obtaining these theorems in elementary fashion may be sketched as follows. First prove (Landau's version of) Bloch's theorem; this is a natural sequel to the lovely one-quarter theorem of Koebe and might well appear at the end of a unit on conformal mapping. Next, derive the Little Picard Theorem from Bloch's theorem. (This much is standard; cf. [**4**, pp. 384–390], [**5**, pp. 98–102], [**8**, pp. 341–346]). Prove Montel's theorem via the heuristic principle (made rigorous) and Picard's little theorem. Finally, derive the Big Picard Theorem and Julia's extension. If desired, Montel's theorem can also be used to give a very brief proof of Schottky's theorem [**4**, pp. 261–262], and Landau's theorem then follows in a couple of lines [**8**, pp. 354–355]. Instructors interested in emphasizing the importance and usefulness of normal families, who find themselves pressed for time and unwilling to tell less than the full truth about the modular function, may find the approach outlined above an attractive alternative to the existing routes.

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SECOND CORRECTION TO: "INNER PRODUCT SPACES"

STANLEY GUDDER AND SAMUEL HOLLAND

In a recent correction [1] the first author pointed out a gap in his paper in this MONTHLY [2]. We now fill that gap. Our argument is a simplified version of that given in [3].

THEOREM. *An inner product space V is complete if every maximal orthonormal set in V is basic.*

Proof. Suppose every maximal orthonormal set in V is basic. We show that (5) of Theorem 3.1 [2] holds. Let f be a nonzero continuous linear functional on V (if $f \equiv 0$, the result is trivial). Let $M = \{x \in V: f(x) = 0\}$. Then M is a closed subspace of V. Let $B = \{x_i : i \in I\}$ be a maximal orthonormal set in M. Extend B to a maximal orthonormal set $B \cup B_1$ of V, where $B_1 = \{y_j : j \in J\}$. Now $J \neq \emptyset$ since otherwise B would be basic in V and then $M = V$ which is a contradiction. Also $y_j \notin M$ for every $j \in J$ since B is maximal in M. Suppose $y_1, y_2 \in B_1$ and let $y = y_1 - f(y_1)y_2/f(y_2)$. Then $y \in M$ and $y \perp x_i$ for every $i \in I$. Since B is maximal in M, $y = 0$ and hence $y_1 = y_2$. It follows that $B_1 = \{y_1\}$. If $x \in V$, then

$$x = \sum \left\langle x, x_i \right\rangle x_i + \left\langle x, y_1 \right\rangle y_1.$$

Hence $f(x) = \left\langle x, y_1 \right\rangle f(y_1) = \left\langle x, \overline{f(y_1)} y_1 \right\rangle$.

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Reply to Query 17. This Query asked for a complete list of programmed text in elementary mathematics. E. G. Begle points out that a bibliography "Programmed Learning and Individually Placed Instruction" can be obtained from Hendershot Programmed Learning, 4114 Ridgewood, Bay City, Michigan 48706.

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