ALGEBRAIC CYCLES AND THE CLASSICAL GROUPS

Part I, Real Cycles

by

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Abstract. The groups of algebraic cycles on complex projective space $\mathbb{P}(V)$ are known to have beautiful and surprising properties. Therefore, when V carries a real structure, it is natural to ask for the properties of the groups of real algebraic cycles on $\mathbb{P}(V)$. Similarly, if V carries a quaternionic structure, one can define quaternionic algebraic cycles and ask the same question. In this paper and its sequel the homotopy structure of these cycle groups is completely determined. It turns out to be quite simple and to bear a direct relationship to characteristic classes for the classical groups.

It is shown, moreover, that certain functors in K-theory extend directly to these groups. It is also shown that, after taking colimits over dimension and codimension, the groups of real and quaternionic cycles carry E_∞-ring structures, and that the maps extending the K-theory functors are E_∞-ring maps. This gives a wide generalization of the results in [BLLMM] on the Segal question.

The ring structure on the homotopy groups of these stabilized spaces is explicitly computed. In the real case it is a simple quotient of a polynomial algebra on two generators corresponding to the first Pontrjagin and first Stiefel-Whitney classes.

These calculations yield an interesting total characteristic class for real bundles. It is a mixture of integral and mod 2 classes and has nice multiplicative properties. The class is shown to be related to the $\mathbb{Z}_2$-equivariant Chern class on Atiyah's KR-theory.

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§1. Introduction.

In recent years a number of results have been proved about the topological groups of algebraic cycles on an algebraic variety X over $\mathbb{C}$. It has been shown for example that when X is projective space, these groups provide useful models for basic classifying spaces in algebraic topology and for certain universal characteristic maps between them. They also yield certain new infinite loop space structures on products of Eilenberg-MacLane spaces which make the total Chern class an infinite loop map. (See [L_2] for a survey.)

Now when X has a real structure, it is natural to consider the *real algebraic cycles* on X. These are simply the cycles defined over $\mathbb{R}$, or equivalently, the cycles fixed by the Galois group $Gal(\mathbb{C}/\mathbb{R})$. When X is projective space $P(V)$ the set of real cycles of codimension-q forms a topological group $\mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}(V))$ whose homotopy-type is independent of V [Lam]. The first main result of this paper is the determination of the topological structure of $\mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}(V))$. We show that it canonically decomposes into a product of Eilenberg-MacLane spaces for the groups $\mathbb{Z}$ and $\mathbb{Z}_2$. (See Theorem 3.3 below.) The resulting structure is rather complicated when compared to the complex case.

Our first explanation for the richness of this structure comes from considering the colimit $\mathcal{Z}^\infty_{\mathbb{R}}$ of these groups over dimension and codimension. Here the algebraic join of cycles induces a ring structure on the homotopy groups and we show that as a ring

$$\pi_*\mathcal{Z}^\infty_{\mathbb{R}} \;\cong\; \mathbb{Z}[x,y]/(2y) \tag{1.1}$$

where x corresponds to the generator of $\pi_4\mathcal{Z}^\infty_{\mathbb{R}} \cong \mathbb{Z}$ and y corresponds to the generator of $\pi_1\mathcal{Z}^\infty_{\mathbb{R}} \cong \mathbb{Z}_2$.

Now the Grassmannian $\mathcal{G}^q(\mathbb{P}(V))$ of codimension-q planes in V includes naturally into $\mathcal{Z}^q(\mathbb{P}(V))$ as degree-1 cycles. Restricting to real points gives an inclusion $\mathcal{G}^q_{\mathbb{R}}(\mathbb{P}(V)) \to \mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}(V))$ which stabilizes to a mapping

$$P : BO_q \;\longrightarrow\; \mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}^\infty).$$

This map represents an interesting total characteristic class which, via Theorem 3.3, is an explicit combination of integral and mod 2 cohomology classes and which has the property that for real vector bundles E and F

$$P(E \oplus F) \;=\; P(E)P(F).$$

In §6 we show that $\mathcal{Z}^\infty_{\mathbb{R}}$ carries the structure of an E_∞-ring space and thus gives rise to an E_∞-ring spectrum. The additive deloopings in this spectrum are the standard deloopings of Eilenberg-MacLane spaces. The multiplicative deloopings extend the product in the group of multiplicative units of the theory. We show that for the multiplicative deloopings, the limiting map

$$P : \mathbf{BO} \;\longrightarrow\; \mathcal{Z}^\infty_{\mathbb{R}}$$

is an infinite loop map yielding a map of spectra $P : \mathfrak{ko} \to \mathfrak{M}_{\mathbb{R}}$ from connective K-theory to the multiplicative spectrum of the theory.

The cycle groups admit two natural homomorphisms:

$$\mathcal{Z}^q_{\mathbb{C}}(\mathbb{P}(V)) \;\longleftarrow\; \mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}(V)) \;\longrightarrow\; \widetilde{\mathcal{Z}}^q_{\mathbb{R}}(\mathbb{P}(V)).$$

The left mapping is the obvious inclusion. The right mapping is projection to the Galois quotient $\widetilde{\mathcal{Z}}^q_{\mathbb{R}}(\mathbb{P}(V)) = \mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}(V))/\mathcal{Z}^q_{\text{av}}(\mathbb{P}(V))$ where $\mathcal{Z}^q_{\text{av}}(\mathbb{P}(V)) = \{c + \overline{c} : c \in \mathcal{Z}^q_{\mathbb{C}}(\mathbb{P}(V))\}$. The colimits of these spaces have E_∞-ring structures for which the limiting maps

$$\mathcal{Z}^\infty_{\mathbb{C}} \longleftarrow \mathcal{Z}^\infty_{\mathbb{R}} \longrightarrow \widetilde{\mathcal{Z}}^\infty_{\mathbb{R}}$$

are E_∞-ring maps. It is known that $\pi_*\mathcal{Z}^\infty_{\mathbb{C}} \cong \mathbb{Z}[s]$ where s corresponds to the generator of $\pi_2\mathcal{Z}^\infty_{\mathbb{C}} \cong \mathbb{Z}$ and $\pi_*\mathcal{Z}^\infty_{\mathbb{R}} \cong \mathbb{Z}_2[y]$ where y corresponds to the generator of $\pi_1\mathcal{Z}^\infty_{\mathbb{R}} \cong \mathbb{Z}_2$. Under the isomorphism (1.1) we show that the maps above induce ring homomorphisms

$$\mathbb{Z}[s] \longleftarrow \mathbb{Z}[x,y]/(2y) \longrightarrow \mathbb{Z}_2[y]$$

given by $x \mapsto s^2$ and $y \mapsto y$.

Composing with the mapping P gives two new mappings

$$\begin{array}{ccccc} & & & & \mathcal{Z}^\infty_{\mathbb{C}} \cong \prod_{k\geq 0} K(\mathbb{Z}, 2k) \\ & & & \nearrow & \\ \mathbf{BO} & \xrightarrow{P} & \mathcal{Z}^\infty_{\mathbb{R}} & & \\ & & & \searrow & \\ & & & & \widetilde{\mathcal{Z}}^\infty_{\mathbb{R}} \cong \prod_{k\geq 0} K(\mathbb{Z}_2, k). \end{array}$$

The top composition classifies the total Chern class of the complexification, and the bottom classifes the total Stiefel-Whitney class. Thus the characteristic class P carries all this information. Furthermore, the maps above all extend to infinite loop maps.

Surprisingly other natural functors in K-theory, such as the forgetful functor, extend from Grassmannians to the spaces of all cycles yielding new proofs of relations between characteristic classes. (See §5.) In §6 these maps are also shown to be infinite loop maps.

There is a unifying perspective on the results discussed above. For this we revisit the map

$$c : \mathbf{BU} \longrightarrow \mathcal{Z}^\infty_{\mathbb{C}} \tag{1.2}$$

and recall that it is a $\mathbb{Z}_2$-map with respect to complex conjugation. Thus we plunge into the world of $\mathbb{Z}_2$-spaces, $\mathbb{Z}_2$-maps, and $\mathbb{Z}_2$-equivariant homotopy theory. Note that a $\mathbb{Z}_2$-space is just a Real space in the sense of Atiyah [A]. Furthermore, $\mathbf{BU}$ is the classifying space for Atiyah's KR-theory. We prove in §6 that $\mathcal{Z}^\infty_{\mathbb{C}}$ has the structure of a $\mathbb{Z}_2 E_\infty$-ring space and that c is a $\mathbb{Z}_2$-equivariant infinite loop map into the multiplicative structure.

In his thesis, Pedro dos Santos has proved that there is a canonical $\mathbb{Z}_2$-equivariant homotopy equivalence

$$\mathcal{Z}^\infty_{\mathbb{C}} \cong \prod_{k\geq 0} K(\underline{\mathbb{Z}}, \mathbb{R}^{n,n}) \tag{1.3}$$

where $K(\underline{\mathbb{Z}}, \mathbb{R}^{n,n})$ denotes the equivariant Eilenberg-MacLane space classifying $\mathbb{Z}_2$-equivariant cohomology indexed at the representation $\mathbb{R}^{n,n} = \mathbb{C}^n$ (with action given by complex conjugation) and with coefficients in the constant Mackey functor $\underline{\mathbb{Z}}$. He furthermore shows that with respect to (1.3) the algebraic join pairing classifies the equivariant cup product and the mapping (1.2) classifies the equivariant total Chern class in KR-theory. Our characteristic mapping P represents the restriction of this equivariant Chern class to the fixed-point sets. (See §6 for details.)

Analogous results for the quaternionic case are proved in Part II of this paper.

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§**2. Spaces of complex cycles.** For expository purposes we quickly review some known results for groups of algebraic cycles over $\mathbb{C}$. The reader is referred to [L$_2$] for an enlarged exposition. Let V be a finite-dimensional complex vector space. For integers $d, q \geq 0$, let $\mathcal{C}^q_d(\mathbb{P}(V))$ denote the Chow variety of effective algebraic cycles of codimension q and degree d in the projective space $\mathbb{P}(V)$. The disjoint union $\mathcal{C}^q(\mathbb{P}(V)) = \coprod_d \mathcal{C}^q_d(\mathbb{P}(V))$ is an abelian topological monoid whose naïve group completion is denoted by $\mathcal{Z}^q(\mathbb{P}(V))$.

As usual let $K(G, n)$ denote the Eilenberg-MacLane space with $\pi_n K(G, n) \cong G$ and $\pi_m K(G, n) \cong 0$ for $m \neq n$, and for a graded abelian group $G_* = \oplus_{j \geq 0} G_j$, let $K(G_*)$ denote the weak product $K(G_*) = \prod_{j \geq 0} K(G_j, j)$.

Theorem 2.1. ([L$_1$]) *For $q \leq \dim \mathbb{P}(V)$ there is a canonical homotopy equivalence*

$$\mathcal{Z}^q(\mathbb{P}(V)) \cong K(\mathbb{Z}, 0) \times K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 4) \times \cdots \times K(\mathbb{Z}, 2q) \tag{2.1}$$

The canonical aspect of this splitting is discussed in Appendix A.

Theorem 2.2. ([LM]) *The algebraic join determines a continuous biadditive pairing*

$$\# : \mathcal{Z}^q(\mathbb{P}(V)) \wedge \mathcal{Z}^{q'}(\mathbb{P}(V')) \longrightarrow \mathcal{Z}^{q+q'}(\mathbb{P}(V \oplus V')) \tag{2.2}$$

which, with respect to the splitting (2.1), represents the cup product.

Theorem 2.3. ([FM]) *Under the join pairing (2.2) the homotopy groups of the limiting space $\mathcal{Z}^\infty$ form a graded ring isomorphic to a polynomial ring in one variable*

$$\pi_* \mathcal{Z}^\infty \cong \mathbb{Z}[s] \tag{2.3}$$

where $s \in \pi_2 \mathcal{Z}^\infty$ is the generator.

If one considers $\mathbb{Z}[s]$ as a graded ring with one generator in degree two, then the quotient $\mathbb{Z}[s]/(s^{q+1})$ has a natural structure of graded abelian group. Using the terminology established above, Theorems 2.1 and 2.3 can be reformulated by saying that one has canonical homotopy equivalences $\mathcal{Z}^q(\mathbb{P}(V)) \cong K\left(\mathbb{Z}[s]/(s^{q+1})\right)$ and $\mathcal{Z}^\infty \cong K(\mathbb{Z}[s])$. Furthermore, the latter equivalence induces, under the join pairing, a ring isomorphism $\pi_*\mathcal{Z}^\infty \cong \mathbb{Z}[s]$.

Let $G^q(\mathbb{P}(V)) = \mathcal{C}_1^q(\mathbb{P}(V))$ denote the Grassmannian of codimension-q planes in $\mathbb{P}(V)$, and let $\mathcal{Z}^q(\mathbb{P}(V))(1)$ denote the connected component of $\mathcal{Z}^q(\mathbb{P}(V))$ consisting of all (not necessarily effective) algebraic cycles of degree 1.

Theorem 2.4. ([LM]) *Under the splitting (2.1) the inclusion*

$$G^q(\mathbb{P}(V)) \hookrightarrow \mathcal{Z}^q(\mathbb{P}(V))(1) \tag{2.4}$$

represents the total Chern class of the tautological q-plane bundle $\xi^q_{\mathbb{C}}$ over $G^q(\mathbb{P}(V))$. Passing to a limit as $\dim(V) \to \infty$ gives a mapping

$$BU_q \longrightarrow \mathcal{Z}^q(\mathbb{P}^\infty)(1) \cong 1 \times \prod_{i=1}^{q} K(\mathbb{Z}, 2i)$$

which classifies the total Chern class of the universal q-plane bundle $\xi^q_{\mathbb{C}}$ over BU_q. Taking the limit as $q \to \infty$ gives a mapping

$$BU \longrightarrow \mathcal{Z}^\infty(1) \cong 1 \times \prod_{i=1}^{\infty} K(\mathbb{Z}, 2i) \overset{\text{def}}{=} K(\mathbb{Z}, 2*) \tag{2.5}$$

which classifies the total Chern class map from K-theory to even cohomology.

This natural presentation of the total Chern class map comes equipped with the following remarkable property.

Theorem 2.5. ([BLLMM]) *The join pairing on $K(\mathbb{Z}, 2*)$ enhances to an infinite loop space structure so that with respect to Bott's infinite loop structure on BU the map (2.5) is an infinite loop map.*

§3. Spaces of real cycles.

A **Real structure** on a complex vector space V is a $\mathbb{C}$-antilinear map $\rho : V \to V$ such that $\rho^2 = 1$. A **Real vector space** is a pair (V, ρ) consisting of a complex vector space V and a Real structure ρ. Any such space is equivalent to $(\mathbb{C}^n, \rho_0)$ where ρ_0 is complex conjugation.

A Real structure ρ on V induces an anti-holomorphic $\mathbb{Z}_2$-action on the complex projective space $\mathbb{P}(V)$ which in turn induces an anti-holomorphic $\mathbb{Z}_2$-action on the Chow varieties $\mathcal{C}_d^q(\mathbb{P}(V))$. This produces an automorphism

$$\rho : \mathcal{Z}^q(\mathbb{P}(V)) \to \mathcal{Z}^q(\mathbb{P}(V)). \tag{3.1}$$

of the topological group of all codimension-q cycles on $\mathbb{P}(V)$.

Definition 3.1. By the **Real algebraic cycles** of codimension q on $\mathbb{P}(V)$ we mean the subgroup $\mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}(V))$ of cycles fixed by the involution ρ. The closed subgroup of Galois sums

$$\mathcal{Z}^q(\mathbb{P}(V))^{av} = \{c + \rho c \mid c \in \mathcal{Z}^q(\mathbb{P}(V))\}$$

is called the group of **averaged cycles**, and the quotient

$$\widetilde{\mathcal{Z}}^q_{\mathbb{R}}(\mathbb{P}(V)) = \mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}(V))/\mathcal{Z}^q(\mathbb{P}(V))^{av}$$

is called the group of **reduced Real algebraic cycles**.

We have adopted the standard definition of real algebraic cycles as those which are fixed by the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Note that the group of reduced cycles is the topological vector space over $\mathbb{Z}_2$ freely generated by the irreducible Real subvarieties of $\mathbb{P}(V)$.

Fix a Real vector space (V, ρ) and let $x_0 = [0 : \cdots : 0 : 1] \in \mathbb{P}(V \oplus \mathbb{C})$. Given an irreducible algebraic subvariety $Z \subset \mathbb{P}(V)$ we define its **algebraic suspension** $\Sigma\!\!\!/ Z = x_0 \# Z \subset \mathbb{P}(V \oplus \mathbb{C})$ to be the union of all projective lines joining Z to x_0. Algebraic suspension extends linearly to a $\mathbb{Z}_2$ equivariant continuous homomorphism

$$\Sigma\!\!\!/ : \mathcal{Z}^q(\mathbb{P}(V)) \to \mathcal{Z}^q(\mathbb{P}(V \oplus \mathbb{C})). \tag{3.2}$$

The Algebraic Suspension Theorem [L$_1$] states that (3.2) is a homotopy equivalence. When V is a Real vector space, T. K. Lam showed that (3.2) is an *equivariant* homotopy equivalence. (See [LLM$_2$] for considerable generalizations.) In particular we have the following.

Theorem 3.2. ([Lam]) *Algebraic suspension induces homotopy equivalences:*

$$\Sigma\!\!\!/ : \mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}(V)) \overset{\cong}{\to} \mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}(V \oplus \mathbb{C})), \qquad \Sigma\!\!\!/ : \mathcal{Z}^q(\mathbb{P}(V))^{av} \overset{\cong}{\to} \mathcal{Z}^q(\mathbb{P}(V \oplus \mathbb{C}))^{av},$$

$$\text{and} \qquad \Sigma\!\!\!/ : \widetilde{\mathcal{Z}}^q_{\mathbb{R}}(\mathbb{P}(V)) \overset{\cong}{\to} \widetilde{\mathcal{Z}}^q_{\mathbb{R}}(\mathbb{P}(V \oplus \mathbb{C})).$$

for all $q < \dim(V)$.

This result shows that the homotopy types of the topological groups $\mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}(V))$, $\mathcal{Z}^q(\mathbb{P}(V))^{av}$ and $\widetilde{\mathcal{Z}}^q_{\mathbb{R}}(\mathbb{P}(V))$ depend only on q, and so we can drop the reference to V. Our first theorems compute these homotopy types.

Theorem 3.3. *There is a canonical homotopy equivalence*

$$\mathcal{Z}^q_{\mathbb{R}} \cong \prod_{n=0}^{q} \prod_{k=0}^{n} K(I_{n,k}, n+k)$$

where

$$I_{n,k} = \begin{cases} 0 & , \text{ if } k \text{ is odd or } k > n; \\ \mathbb{Z} & , \text{ if } k = n \text{ and } k \text{ is even}; \\ \mathbb{Z}_2 & , \text{ if } k < n \text{ and } k \text{ is even}. \end{cases}$$

Theorem 3.4. *There is a canonical homotopy equivalence*

$$\mathcal{Z}^q_{av} \;\cong\; \prod_{n=0}^{q}\prod_{k=0}^{n} K\left(I^{av}_{n,k},\; n+k\right)$$

where $I^{av}_{0,0} = 2\mathbb{Z}$ *and for* $n+k>0$

$$I^{av}_{n,k} \;=\; \tilde{H}_{k-1}(\mathbb{P}^{n-1}_{\mathbb{R}};\mathbb{Z}) \;=\; \begin{cases} 0 & \text{, if } k \text{ is odd or } k>n; \\ \mathbb{Z} & \text{, if } k=n \text{ and } k\geq 2 \text{ is even;} \\ \mathbb{Z}_2 & \text{, if } k<n \text{ and } k\geq 2 \text{ is even.} \end{cases}$$

The homomorphism on homotopy groups induced by the inclusion $\mathcal{Z}^q_{av} \subset \mathcal{Z}^q_{\mathbb{R}}$ *is injective, and with respect to the splittings above, it maps* $I^{av}_{n,k}$ *to* $I_{n,k}$ *in the obvious way. (This explains the* $2\mathbb{Z}$ *in* $I_{0,0}$*.)*

Theorem 3.5. ([Lam]) *There is a canonical homotopy equivalence*

$$\widetilde{\mathcal{Z}}^q_{\mathbb{R}} \;\cong\; K(\mathbb{Z}_2,0)\times K(\mathbb{Z}_2,1)\times K(\mathbb{Z}_2,2)\times\cdots\times K(\mathbb{Z}_2,q). \tag{3.1}$$

The proofs of these results are given in §8. Useful diagrams of the graded groups $\mathcal{I}^{av}_{*,*}$ and $\mathcal{I}_{*,*}$ are given in §9.

§4. The ring structure.

The homotopy groups

$$\pi_*\mathcal{Z}^q_{\mathbb{R}} \;=\; \bigoplus_{0\leq k\leq n\leq q} I_{n,k} \tag{4.1}$$

are vastly simplified conceptually if one takes into account their **multiplicative structure**. The algebraic join pairing (2.2) restricts to a pairing

$$\# : \mathcal{Z}^q_{\mathbb{R}} \wedge \mathcal{Z}^{q'}_{\mathbb{R}} \;\longrightarrow\; \mathcal{Z}^{q+q'}_{\mathbb{R}}$$

which gives $\pi_*\mathcal{Z}^\infty_{\mathbb{R}}$ the structure of a commutative ring. Since the join of an averaged cycle with a fixed cycle is again an averaged cycle, the subgroup $\pi_*\mathcal{Z}^\infty_{av}$ is an *ideal* in this ring.

In §9 we will prove the following result.

Theorem 4.1. *There is a ring isomorphism*

$$\pi_*\mathcal{Z}^\infty_{\mathbb{R}} \;\cong\; \mathbb{Z}[x,y]/(2y) \tag{4.2}$$

where x *corresponds to the generator of* $\pi_4\mathcal{Z}^\infty_{\mathbb{R}} \cong \mathbb{Z}$ *and* y *corresponds to the generator of* $\pi_1\mathcal{Z}^\infty_{\mathbb{R}} \cong \mathbb{Z}_2$*, and where* $(2y)$ *denotes the principal ideal generated by* $2y$ *in the polynomial ring* $\mathbb{Z}[x,y]$*. Under this isomorphism the ideal* $\pi_*\mathcal{Z}^\infty_{av} \subset \pi_*\mathcal{Z}^\infty_{\mathbb{R}}$ *corresponds to the ideal*

$$\pi_*\mathcal{Z}^\infty_{av} \;\cong\; (2,x)$$

generated by 2 and x*. Furthermore, with respect to the isomorphisms (4.1) and (4.2), we have*

$$I_{2m+\ell,2m} \quad \text{is the cyclic subgroup generated by} \quad x^m y^\ell.$$

Corollary 4.2. *The algebraic join induces a ring structure on $\pi_*\widetilde{\mathcal{Z}}^\infty_{\mathbb{R}}$. There is a canonical ring isomorphism*

$$\pi_*\widetilde{\mathcal{Z}}^\infty_{\mathbb{R}} \;\cong\; \mathbb{Z}_2[y]$$

where y in the generator of $\pi_1\widetilde{\mathcal{Z}}^\infty_{\mathbb{R}} = \mathbb{Z}_2$.

Remark 4.3. Consider the polynomial ring $\mathbb{Z}[x,y]$ on the variables x and y, of degrees 4 and 1, respectively. Given a non-negative integer q, define the ideal

$$\mathcal{J}_q = (\ 2y,\ \{x^m y^j\ :\ 2m+j=q+1\ \}) \subset \mathbb{Z}[x,y],$$

and denote $\mathcal{J}_\infty = (2y)$. Each quotient ring $\mathcal{R}^q_* = \mathbb{Z}[x,y]/\mathcal{J}_q$, $q=0,\dots,\infty$, has the natural structure of a graded abelian group.

Using this notation, Theorem 3.3 can be rephrased by saying that there is a canonical equivalence

$$\mathcal{Z}^q_{\mathbb{R}} \cong K(\mathcal{R}^q_*).$$

Under this equivalence the direct summand $I_{2q+j,2q}$ of the $(4q+j)$-th homotopy group of $\mathcal{Z}^q_{\mathbb{R}}$ is precisely the subgroup of $\mathcal{R}^q_*$ generated by $x^q y^j$. One can rephrase Theorem 3.4 and Corollary 3.5 in a similar fashion.

We also prove that there are canonical equivalences

$$\mathcal{Z}^\infty_{\mathbb{R}} \;\cong\; K(\ \mathcal{R}^\infty_*\ ),\quad \widetilde{\mathcal{Z}}^\infty_{\mathbb{R}} \;\cong\; K(\ \mathbb{Z}_2[y]\ )\quad \text{and}\quad \mathcal{Z}^\infty_{av} \cong K(\ I^{av}\ ).$$

Here I^{av} is the ideal $I^{av} = (2,x) \subset \mathcal{R}^\infty_*$. Furthermore, these homotopy equivalences induce the ring isomorphisms presented in Theorems 4.1 and Corollary 4.2.

§**5. Extending functors from K-theory.** We shall now show that certain basic functors in classical representation theory carry over to algebraic cycles. This remarkable fact together with [LM] and the results of §3 leads to a new proof of the basic relationships among characteristic classes.

Before beginning we set some notation. For all $k \geq 0$ let

$$\iota_{2k} \in H^{2k}(K(\mathbb{Z},2k);\mathbb{Z}) \cong \mathbb{Z} \qquad \text{and} \qquad \widetilde{\iota}_k \in H^k(K(\mathbb{Z}_2,k);\mathbb{Z}_2) \cong \mathbb{Z}_2$$

denote the *fundamental classes* (i.e., the canonical generators). Let c_k, w_k, and p_k denote respectively the k$^{\text{th}}$ Chern, Stiefel-Whitney, and Pontrjagin classes.

Complexification. Consider a Real vector space (V,ρ) and the map $(V,\rho) \to V$ which forgets the Real structure. Associated to this is the homomorphism $\mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}(V)) \hookrightarrow \mathcal{Z}^q_{\mathbb{C}}(\mathbb{P}(V))$ which simply includes the subgroup fixed by ρ into the group of all cycles. Restricting to linear cycles gives a commutative diagram

$$\begin{array}{ccc} G^q_{\mathbb{R}}(\mathbb{P}(V)) & \longrightarrow & G^q_{\mathbb{C}}(\mathbb{P}(V)) \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle c} \\ \mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}(V)) & \longrightarrow & \mathcal{Z}^q_{\mathbb{C}}(\mathbb{P}(V)) \end{array} \tag{5.1}$$

where

$$G^q_{\mathbb{R}}(\mathbb{P}(V)) \;=\; \{\ell \in G^q_{\mathbb{C}}(\mathbb{P}(V)) \,:\, \rho(\ell) = \ell\}$$

is the Grassmannian of real subspaces of codimension-q in $V_{\mathbb{R}} = \{v \in V \,:\, \rho(v) = v\}$.

Recall from 2.4 that under the canonical identification $\mathcal{Z}^q_{\mathbb{C}}(\mathbb{P}(V)) \;\cong\; \prod_{k=0}^q K(\mathbb{Z}, 2k)$ the map c in (5.1) classifies the total Chern class of the tautological q-plane bundle $\xi^q_{\mathbb{C}} \longrightarrow G^q_{\mathbb{C}}(\mathbb{P}(V))$, i.e.,

$$c^*(\iota_{2k}) = c_k(\xi^q_{\mathbb{C}}) \qquad \text{for } k = 0, ..., q.$$

Consider the composition

$$w = \pi \circ P : G^q_{\mathbb{R}}(\mathbb{P}(V)) \longrightarrow \widetilde{\mathcal{Z}}^q_{\mathbb{R}}(\mathbb{P}(V)) \tag{5.2}$$

where $\pi \,:\, \mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}(V)) \to \widetilde{\mathcal{Z}}^q_{\mathbb{R}}(\mathbb{P}(V)) \;=\; \mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}(V))/\mathcal{Z}^q(\mathbb{P}(V))^{\mathrm{av}}$ is the projection. It is a result of Lam [Lam] that under the canonical identification $\widetilde{\mathcal{Z}}^q_{\mathbb{R}}(\mathbb{P}(V)) \;\cong\; \prod_{k=0}^q K(\mathbb{Z}_2,\, k)$, the map w classifies the total Stiefel-Whitney class of the tautological real q-plane bundle $\xi^q_{\mathbb{R}} \longrightarrow G^q_{\mathbb{R}}(\mathbb{P}(V))$, i.e.,

$$w^*(\widetilde{\iota}_k) = w_k(\xi^q_{\mathbb{R}}) \qquad \text{for } k = 0, ..., q.$$

We now set $V = \mathbb{C}^n$ and take the colimit of the spaces in (5.1) and (5.2) as $n \to \infty$. This gives a diagram

$$\begin{array}{ccc} BO_q & \xrightarrow{\ \gamma\ } & BU_q \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle c} \\ \mathcal{Z}^q_{\mathbb{R}} & \xrightarrow{\ \Gamma\ } & \mathcal{Z}^q_{\mathbb{C}} \\ {\scriptstyle \pi}\downarrow & & \\ \widetilde{\mathcal{Z}}^q_{\mathbb{R}} & & \end{array} \tag{5.3}$$

where $\mathcal{Z}^q_{\mathbb{C}} \equiv \lim\limits_{n\to\infty} \mathcal{Z}^q_{\mathbb{C}}(\mathbb{P}(\mathbb{C}^n))$, etc.. By using (3.3), this can be canonically rewritten as

$$\begin{array}{ccc} BO_q & \xrightarrow{\ \gamma\ } & BU_q \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle c} \\ \prod\limits_{n=0}^{[q/2]} \prod\limits_{i=1}^{q-2n} K(\mathbb{Z}_2, 4n+i) \times \prod\limits_{k=0}^{[q/2]} K(\mathbb{Z}, 4k) & \xrightarrow{\ \Gamma\ } & \prod\limits_{k=0}^{q} K(\mathbb{Z}, 2k) \\ {\scriptstyle \pi}\downarrow & & \\ \prod\limits_{k=0}^{q} K(\mathbb{Z}_2, k) & & \end{array} \tag{5.4}$$

The map γ on classifying spaces is the one induced by the inclusion $O_q \subset U_q$ associated to the complexification of vector spaces $V_{\mathbb{R}} \mapsto V_{\mathbb{R}} \otimes \mathbb{C}$.

Consider the classes

$$j_{2k} \stackrel{\text{def}}{=} \Gamma^* \iota_{2k} \in H^{2k}(\mathcal{Z}^q_{\mathbb{R}}; \mathbb{Z}) \qquad \text{and} \qquad \widetilde{j}_k \stackrel{\text{def}}{=} \pi^* \widetilde{\iota}_k \in H^k(\mathcal{Z}^q_{\mathbb{R}}; \mathbb{Z}_2).$$

From these theorems and the commutativity of the diagrams above we see that

$$(5.5) \qquad (-1)^k P^* j_{4k} = p_k(\xi^q_{\mathbb{R}}) \qquad \text{and} \qquad P^* \widetilde{j}_k = w_k(\xi^q_{\mathbb{R}}).$$

In particular, j_{4k} is not divisible and not torsion, whereas j_{4k+2} has order 2. From the factoring (5.10), (5.11) below we see that $j_{4k} = \iota_{4k} + \tau$ where $2\tau = 0$.

The forgetful functor. For a complex vector space V one constructs the conjugate space $\overline{V}$ by taking the same additive group and defining a new scalar multiplication $\bullet$ by $t \bullet v \equiv \bar{t} v$. With this we can associate to V a Real space $([V]_{\mathbb{R}}, \rho)$ where

$$[V]_{\mathbb{R}} = V \oplus \overline{V} \qquad \text{and} \qquad \rho(v, w) = (w, v).$$

For any $q < \dim(V)$ we have a map

$$\Phi : \mathcal{Z}^q_{\mathbb{C}}(\mathbb{P}(V)) \longrightarrow \mathcal{Z}^{2q}_{\mathbb{R}}(\mathbb{P}(V \oplus \overline{V}))$$

defined by

$$\Phi(c) = c \# c$$

where $\#$ is the complex join. This construction gives commutative diagrams

$$\begin{array}{ccc} G^q_{\mathbb{C}}(\mathbb{P}(V)) & \xrightarrow{\phi} & G^{2q}_{\mathbb{R}}(\mathbb{P}([V]_{\mathbb{R}})) \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle P} \\ \mathcal{Z}^q_{\mathbb{C}}(\mathbb{P}(V)) & \xrightarrow{\Phi} & \mathcal{Z}^{2q}_{\mathbb{R}}(\mathbb{P}([V]_{\mathbb{R}})) \end{array}$$

which stabilize as above to commutative diagrams

$$\begin{array}{ccc} BU_q & \xrightarrow{\phi} & BO_{2q} \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle P} \\ \mathcal{Z}^q_{\mathbb{C}} & \xrightarrow{\Phi} & \mathcal{Z}^{2q}_{\mathbb{R}}. \end{array}$$

Note that ϕ is the map induced by the standard inclusion $U_q \subset O_{2q}$.

Relations. Consider the diagram

$$(5.6)\qquad \begin{array}{ccccc} BU_q & \longrightarrow & BO_{2q} & \longrightarrow & BU_{2q} \\ {\scriptstyle c}\downarrow & & {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle c} \\ \prod_{j=0}^{q} K(\mathbb{Z},2j) & \xrightarrow{\Phi} & \mathcal{Z}_{\mathbb{R}}^{2q} & \xrightarrow{\Gamma} & \prod_{j=0}^{2q} K(\mathbb{Z},2j) \end{array}$$

Note that if V has a real structure ρ, then under the isomorphism $I \oplus \rho : V \oplus \overline{V} \longrightarrow V \oplus V$, the map $\Phi : \mathcal{Z}_{\mathbb{C}}^{q} \to \mathcal{Z}_{\mathbb{R}}^{2q}$ becomes $\Phi(c) = c\#\rho_*(c)$. It follows that

$$\Gamma \circ \Phi(c) = c\#\rho_*(c)$$

for $c \in \mathcal{Z}_{\mathbb{C}}^{q}$. We conclude the following.

Proposition 5.1. *The composition $\Gamma \circ \Phi$ satisfies*

$$(5.7)\qquad \begin{aligned} (\Gamma\circ\Phi)^*\iota_{2k} &= \sum_{i+j=k} (-1)^j \iota_{2i} \cup \iota_{2j} \\ &= \begin{cases} 2\sum_{j=0}^{m-1} (-1)^j \iota_{2j} \cup \iota_{2(2m-j)} + (-1)^m \iota_{2m}^2 & \text{if } k = 2m \\ 0 & \text{if } k = 2m+1 \end{cases} \end{aligned}$$

for all k.

Proof. By Theorem 2.2 the join mapping $\# : \mathcal{Z}_{\mathbb{C}}^{q} \times \mathcal{Z}_{\mathbb{C}}^{q} \longrightarrow \mathcal{Z}_{\mathbb{C}}^{2q}$ has the characterizing property that $\#^*\iota_{2k} = \sum_{i+j=k} \iota_{2i} \otimes \iota_{2j}$. It is straightforward to verify that the map $\rho : \mathcal{Z}_{\mathbb{C}}^{q} \to \mathcal{Z}_{\mathbb{C}}^{q}$, induced by the real structure ρ, has the characterizing property that $\rho^*\iota_{2k} = (-1)^k\iota_{2k}$. Taking the composition $\mathcal{Z}_{\mathbb{C}}^{q} \xrightarrow{\Delta} \mathcal{Z}_{\mathbb{C}}^{q} \times \mathcal{Z}_{\mathbb{C}}^{q} \xrightarrow{1\times\rho} \mathcal{Z}_{\mathbb{C}}^{q} \times \mathcal{Z}_{\mathbb{C}}^{q} \xrightarrow{\#} \mathcal{Z}_{\mathbb{C}}^{q}$ and pulling back ι_{2k} gives the result. $\square$

Similarly we have the diagram

$$(5.8)\qquad \begin{array}{ccccc} BO_q & \xrightarrow{\gamma} & BU_q & \xrightarrow{\phi} & BO_{2q} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{Z}_{\mathbb{R}}^{q} & \xrightarrow{\Gamma} & \prod_{j=0}^{q} K(\mathbb{Z},2j) & \xrightarrow{\Phi} & \mathcal{Z}_{\mathbb{R}}^{2q} \end{array}$$

and the relation

$$\Phi \circ \Gamma(c) = c\#\rho_*c = c\#c,$$

i.e., $\Phi \circ \Gamma$ is just the **squaring map**. Thus $\Phi \circ \Gamma$ induces a map

$$\widetilde{\Phi \circ \Gamma} : \widetilde{\mathcal{Z}}_{\mathbb{R}}^{q} \longrightarrow \widetilde{\mathcal{Z}}_{\mathbb{R}}^{2q}$$

which is also the squaring map.

Proposition 5.2. *Under the canonical equivalence 3.5, the composition $\Phi \circ \Gamma$ satisfies*

$$(5.9) \qquad (\widetilde{\Phi \circ \Gamma})^* \widetilde{\iota}_k \;=\; \sum_{i+j=k} \widetilde{\iota}_i \cup \widetilde{\iota}_j \;=\; \begin{cases} \widetilde{\iota}_m^2 & \text{if } k = 2m \\ 0 & \text{if } k = 2m+1 \end{cases}$$

for all k.

Proof. Using the fact that the join $\# : \widetilde{\mathcal{Z}}^q_{\mathbb{R}} \times \widetilde{\mathcal{Z}}^q_{\mathbb{R}} \longrightarrow \widetilde{\mathcal{Z}}^{2q}_{\mathbb{R}}$ classifies the cup product [Lam], one proceeds as in the proof of 5.1. □

Note that the composition $BO_q \to \widetilde{\mathcal{Z}}^q_{\mathbb{R}} \to \widetilde{\mathcal{Z}}^{2q}_{\mathbb{R}}$ classifies the square of the total Stiefel-Whitney class $w(\xi^q_{\mathbb{R}})^2 = \sum_{k=0}^q w_k(\xi^q_{\mathbb{R}})^2$.

Notice that as q increases the diagrams (5.8) are included in one another. From [L$_1$] and [Lam] we know that if we define $\mathcal{Z}^{q-1}_{\mathbb{C}} \subset \mathcal{Z}^q_{\mathbb{C}}$ via the inclusion $V \times \{0\} \subset V \oplus \mathbb{C}$, then

$$\mathcal{Z}^q_{\mathbb{C}} / \mathcal{Z}^{q-1}_{\mathbb{C}} \cong K(\mathbb{Z}, 2q) \qquad \text{and} \qquad \widetilde{\mathcal{Z}}^q_{\mathbb{R}} / \widetilde{\mathcal{Z}}^{q-1}_{\mathbb{R}} \cong K(\mathbb{Z}_2, q).$$

From this we obtain a diagram

$$(5.10) \qquad \begin{array}{ccc} BO_q/BO_{q-1} & \longrightarrow & BU_q/BU_{q-1} \\ \downarrow & & \downarrow \\ \mathcal{Z}^q_{\mathbb{R}}/\mathcal{Z}^{q-1}_{\mathbb{R}} & \xrightarrow{\Gamma_0} & K(\mathbb{Z}, 2q) \\ \pi_0 \downarrow & & \\ K(\mathbb{Z}_2, q). & & \end{array}$$

and from Theorem 3.3 we know that

$$(5.11) \qquad \begin{aligned} \mathcal{Z}^{2q_0}_{\mathbb{R}} / \mathcal{Z}^{2q_0-1}_{\mathbb{R}} &= K(\mathbb{Z}, 4q_0) \times \prod_{i=0}^{q_0-1} K(\mathbb{Z}_2, 2q_0 + 2i) \qquad \text{and} \\ \mathcal{Z}^{2q_0+1}_{\mathbb{R}} / \mathcal{Z}^{2q_0}_{\mathbb{R}} &= \prod_{i=0}^{q_0} K(\mathbb{Z}_2, 2q_0 + 2i + 1). \end{aligned}$$

By Theorem 3.4 the map π_0 kills all factors with $i > 0$. However, Γ_0 could represent non-trivial cohomology operations on $K(\mathbb{Z}_2, 2*)$.

At this point one might naturally ask: What is the cohomology class $\Gamma_0^*(\iota_{2q})$? The following answer, which was provided by the referee, will be proved at the end of §8.

Proposition 5.3. *The class $\Gamma_0^*(\iota_{2q})$ is is given by*

$$\Gamma_0^*(\iota_{2q}) \;=\; \begin{cases} \iota_{2q}\otimes 1\otimes 1\cdots\otimes 1 & \text{when } q \text{ is even} \\ 1\otimes\cdots\otimes 1\otimes\beta(\widetilde{\iota}_{2q-1}) & \text{when } q \text{ is odd} \end{cases}$$

where β is the Boskstein operator.

Consider now the composition P given by

$$BO_q \xrightarrow{\;Q\;} BO_q/BO_{q-1} \xrightarrow{\;P_0\;} \mathcal{Z}^q_{\mathbb{R}}/\mathcal{Z}^{q-1}_{\mathbb{R}}$$

where Q is the quotient and P_0 comes from (5.10). The image of Q^* in $H^*(BO_q;\mathbb{Z}_2)\cong\mathbb{Z}_2[w_1,\ldots,w_q]$ is the ideal (w_q) generated by the q^{th} Stiefel-Whitney class w_q. Consider the canonical product structure (5.11) and let $K(\mathbb{Z}_2;x^iy^{q-2i})$ denote the Eilenberg-MacLane space $K(\mathbb{Z}_2;q+2i)$ corresponding to the monomial x^iy^{q-2i} under the identification $\mathcal{Z}_{\mathbb{R}}\cong K\left(\mathbb{Z}[x,y]/(2y)\right)$. With this notation, let $\widetilde{\iota}_{q,2i}$ denote the fundamental class in $H^{q+2i}(K(\mathbb{Z}_2,x^iy^{q-2i});\mathbb{Z}_2)$ pulled back to the product. Note that $\widetilde{\iota}_{q,2i}$ is the Kronecker dual to the class $\theta_{q,2i}$ introduced in (9.11). Then we see that

$$P^*\widetilde{\iota}_{q,2i} \;=\; F_{q,2i}(w_1,\ldots,w_{q-1})\cdot w_q \tag{5.12}$$

where $F_{q,2i}(\xi_1,\ldots,\xi_{q-1})\in\mathbb{Z}_2[\xi_1,\ldots,\xi_{q-1}]$ is a homogeneous polynomial of weighted degree $2i$, i.e., $F_{q,2i}(t\xi_1,t^2\xi_2,\ldots,t^{q-1}\xi_{q-1})=t^{2i}F_{q,2i}(\xi)$. These polynomials determine P up to homotopy.

§**6. Equivariant infinite loop space structures and KR-theory.** In this section we shall show that our spaces of complex algebraic cycles have the structure of an **equivariant E_∞-ring spaces** (cf. [LMS]), under the $\mathbb{Z}_2$ action induced by complex conjugation. The principle is the same as in [LLM$_1$], where the ruled join of cycles induces the infinite loop structure. However, here we obtain $RO(\mathbb{Z}_2)$-graded cohomology theories, as opposed to $R(\mathbb{Z}_2)$-graded ones.

Furthermore, we show that one obtains two canonical equivariant infinite loop spaces from these constructions. The first one comes from delooping the additive structure, which yields an equivariant ring spectrum. The second one come from delooping the multiplicative units of the original ring space. This yields an equivariant spectrum which is directly related to characteristic classes in Atiyah's KR-theory.

It follows from these constructions that the space of real cycles $\mathcal{Z}^\infty_{\mathbb{R}}$ is also an E_∞-ring space and that most of the maps introduced in previous sections are maps of E_∞-ring spaces. Our arguments involve P. May's use of equivariant $\mathcal{I}_*$-functors and make extensive use of the constructions in [LLM$_1$]. We shall briefly introduce the concepts but refer the reader to [LLM$_1$] for many details.

Consider $\mathbb{C}^\infty$ as a direct sum $\mathbb{R}^\infty\oplus i\mathbb{R}^\infty$ with its standard orthogonal inner product, and where $\mathbb{Z}_2$ acts by complex conjugation. Then $\mathbb{C}^\infty$ contains infinitely many copies of each irreducible real representation of $\mathbb{Z}_2$, in other words, in the terminology of [LMS] it is a *complete* $\mathbb{Z}_2$*-universe*. It will be fixed throughout this discussion.

In general, suppose G is a finite group and let $\mathcal{U}$ be a fixed G-universe. Recall that an **equivariant** infinite loop space X, indexed on $\mathcal{U}$, is a based G-space for which there is collection of G-spaces $\{X(V) \mid V \subset \mathcal{U} \text{ is a } G\text{-submodule}\}$ together with G-equivariant homeomorphisms $X \cong \Omega^V X(V)$. Here $\Omega^V X(V)$ denotes the space of based maps $F(S^V, X(V))$ from the one-point compactification S^V of V to $X(V)$, and $\Omega^V X(V)$ is equipped with its natural structure of G-space. The structural homeomorphisms are coherent in the sense that, if for a given submodule $W \subset V$ one denotes by $V - W$ the orthogonal complement of W in V, then there are compatible G-homeomorphisms $X(W) \cong \Omega^{V-W} X(V)$. In general, to give a G-space X' an equivariant infinite loop space structure is to provide a G-homotopy equivalence between X' and an equivariant infinite loop space X.

Remark 6.1. If $\{X(V) \mid V \subset \mathcal{U}\}$ is the collection of equivariant "deloopings" of the equivariant infinite loop space X, let $X(n)$ denote $X(\mathbb{R}^n)$ for the trivial G-module $\mathbb{R}^n$. Then for any subgroup $H \leq G$, the fixed point set X^H has the structure of a (non-equivariant) infinite loop space, since the G-homeomorphism $X \cong \Omega^n X(n)$ gives a homeomorphism of fixed point sets $X^H \cong (\Omega^n X(n))^H = \Omega^n \left(X(n)^H\right)$. Furthermore, if $H \leq K \leq G$ are subgroups then, under the structure defined above, the inclusion $X^K \subset X^H$ is obviously a map of (non-equivariant) infinite loop spaces.

In order to show that a G-space X has the structure of an equivariant infinite loop space, we use the machinery developed in [CW]. In this formulation, one considers the category of $G\mathcal{L}(\mathcal{U})$**-spaces**, whose objects are G-spaces on which there is an action of the equivariant linear isometries operad $G\mathcal{L}(\mathcal{U})$ (cf. [M_3, pp 10 ff], [CW]), and where a **map of** $G\mathcal{L}(\mathcal{U})$**-spaces** is a G-map which commutes with the action of $G\mathcal{L}(\mathcal{U})$. The next result is a formulation of the main results from [CW], suitable for our purposes.

Theorem 6.2. ([CW]) *Let $\mathcal{U}$ be a complete G-universe and let X be a $G\mathcal{L}(\mathcal{U})$-space which is G-group-complete. In other words, for each subgroup $K \leq G$ the induced $\mathcal{H}$-space structure makes $\pi_0(X^K)$ a group. Then X has an equivariant infinite loop space structure. This structure is natural in the sense that any map of G-group-complete $G\mathcal{L}(\mathcal{U})$-spaces induces an equivariant infinite loop map.*

From now on, we restrict ourselves to the case where $G = \mathbb{Z}_2$ and fix the $\mathbb{Z}_2$ universe $\mathcal{U} = \mathbb{C}^\infty$ described above. For simplicity we shatll write $\mathcal{L}$ instead of $\mathbb{Z}_2\mathcal{L}(\mathcal{U})$, and we shall avoid mentioning the universe in most instances.

Following Atiyah's terminology [A], define a **Real topological space** to be a pair (X, ρ) where X is a space and $\rho : X \to X$ is an involution. In other words, X is a $\mathbb{Z}_2$-space. A **Real mapping** $f : X \to Y$ between Real spaces is one which commutes with the involutions (a $\mathbb{Z}_2$-equivariant map). We denote by $\mathbb{Z}_2\mathcal{T}$ the category of compactly generated, based Hausdorff Real topological spaces, with base-point fixed by the action. The morphism spaces in $\mathbb{Z}_2\mathcal{T}$ are given the usual topology in the compactly generated category, and have the natural $\mathbb{Z}_2$-action on them.

A natural way of constructing actions of the equivariant linear isometries operad $\mathcal{L}$ uses the following notions. Let $\mathbb{Z}_2\mathcal{I}_*$ the subcategory of the category of finite dimensional hermitian $\mathbb{Z}_2$-modules and $\mathbb{Z}_2$-module morphisms, whose morphisms are also linear isometries.

Definition 6.3. A **$\mathbb{Z}_2\mathcal{I}_*$-space** (or **$\mathbb{Z}_2\mathcal{I}_*$-functor**) (T,ω) is a continuous covariant functor $T:\mathbb{Z}_2\mathcal{I}_*\longrightarrow\mathbb{Z}_2\mathcal{T}$ together with a (coherently) commutative, associative and continuous natural transformation $\omega:T\times T\longrightarrow T\circ\oplus$ such that

(1) If $x\in TV$ and if $1\in T\{0\}$ is the basepoint, then

$$\omega(x,1)=x\in T(V\oplus\{0\})=TV,$$

(2) If $V=V'\oplus V''$, then the map $TV'\longrightarrow TV$ given by $x\mapsto\omega(x,1)$ is a homeomorphism onto a closed subset;
(3) Each sum map $\omega:T(V)\times T(W)\to T(V\oplus W)$ is a G-map;
(4) Each evaluation map $e:\mathbb{Z}_2\mathcal{I}_*(V,W)\times T(V)\to T(W)$ is a G-map.

The following result is a direct consequence of the techniques in [M$_3$]. See the discussion in [LLM$_1$, §2].

Theorem 6.4. *If (T,ω) is an $\mathbb{Z}_2\mathcal{I}_*$-space, then*

$$T(\mathbb{C}^\infty)\ =\ \lim_{V\subset\mathbb{C}^\infty}T(V),$$

where the limit is taken over finite-dimensional $\mathbb{Z}_2$-submodules of $\mathbb{C}^\infty$, is an $\mathcal{L}$-space. Any map $\Phi:(T,\omega)\longrightarrow(T',\omega')$ of $\mathcal{I}_*$-spaces, induces a mapping $\Phi:T(\mathbb{C}^\infty)\longrightarrow T'(\mathbb{C}^\infty)$ of $\mathcal{L}$-spaces.

A given $V\in\mathbb{Z}_2\mathcal{I}_*$ can be written as $V=\mathbb{R}^n\oplus\sigma\otimes\mathbb{R}^m$, where $\mathbb{R}^k$ denotes a trivial representation of rank k and σ is the sign representation of $\mathbb{Z}_2$. In particular, if one denotes by $V_{\mathbb{R}}$ the underlying real vector space of V, then the sum $V\oplus\sigma\otimes V$ is canonically isomorphic to $V_{\mathbb{C}}\overset{\text{def}}{=}V_{\mathbb{R}}\otimes\mathbb{C}$ as a $\mathbb{Z}_2$-module, where the action on the latter is given by complex conjugation. Given such V, we denote its real dimension by $v=n+m$, and for any map $f:V\to W$ we denote by $f_{\mathbb{C}}$ its natural extension to the complexified vector spaces.

Example 6.5. (The Grassmann functor) Given $V\in\ \mathbb{Z}_2\mathcal{I}_*$ of dimension v, let $T_G(V)=G^v(V_{\mathbb{C}}\oplus V_{\mathbb{C}})=G^v(\mathbb{P}(V_{\mathbb{C}}\oplus V_{\mathbb{C}}))$ be the Grassmannian of codimension-v complex planes in $V_{\mathbb{C}}\oplus V_{\mathbb{C}}$, with distinguished point $1_G=V_{\mathbb{C}}\oplus\{0\}$. To a linear isometric embedding $f:V\to W$ we define $T_Gf:T_GV\to T_GW$ on a plane $P\subset V_{\mathbb{C}}\oplus V_{\mathbb{C}}$ by $T_Gf(P)=((f_{\mathbb{C}}V_{\mathbb{C}})^\perp\oplus\{0\})\oplus(f_{\mathbb{C}}\oplus f_{\mathbb{C}})(P)$. The natural transformation $\omega_G:T_G\times T_G\longrightarrow T_G\circ\oplus$ is given by the direct sum, i.e., for $P\in T_GV$ and $P'\in T_GV'$, we define $\omega_G(P,P')=\tau_*(P\oplus P')$ where $\tau:V\oplus V\oplus V'\oplus V'\longrightarrow V\oplus V'\oplus V\oplus V'$ is the isometry interchanging the middle factors. This is an $\mathbb{Z}_2\mathcal{I}_*$-functor, and

$$T_G(\mathbb{C}^\infty)\ =\ \mathbf{BU}$$

is then a $\mathbb{Z}_2$-equivariant $\mathcal{L}$-space, and hence it is an equivariant infinite loop space; cf. Theorem 6.2. According to Remark 6.1, if $\{0\}$ denotes the trivial subgroup of $\mathbb{Z}_2$, then both fixed point sets $\mathbf{BU} = \mathbf{BU}^{\{0\}}$ and $\mathbf{B0} = \mathbf{BU}^{\mathbb{Z}_2}$ inherit infinite loop space structures which makes the canonical "complexification" inclusion $\mathbf{BO} \hookrightarrow \mathbf{BU}$ a map of infinite loop spaces. These are the standard Bott infinite loop space structures. (See [M$_3$, pg.16].)

This equivariant structure on $\mathbf{BU}$ classifies an $RO(\mathbb{Z}_2)$-graded equivariant cohomology theory which we now recall.

Definition 6.6. Let (X, ρ) be a Real space. A **Real vector bundle** over (X, ρ) is a Real space (E, ρ_E) where $\pi : E \to X$ is a complex vector bundle, ρ_E is a complex *anti-linear* bundle map, and π is a Real map, i.e., $\pi\rho_E = \rho\pi$.

A Real projective variety with its complex conjugation involution gives a Real space. Important examples are the Grassmannians $G^q(\mathbb{C}^n)$ and the Chow varieties. The universal q-plane bundle ξ^q over $G^q(\mathbb{C}^n)$ is a Real bundle.

Proposition 6.7. *Let (X, ρ) be a Real space which is compact and Hausdorff. Then the association $f \mapsto f^*\xi^q$ gives an equivalence of functors:*

$$[X, G^q(\mathbb{C}^\infty)]_{\mathbb{R}} \xrightarrow{\cong} \mathrm{V}ect^q_{\mathbb{R}}(X),$$

from homotopy classes of Real mappings $X \to G^q(\mathbb{C}^\infty)$ to the set $\mathrm{V}ect^q_{\mathbb{R}}(X)$ of equivalence classes of Real q-dimensional vector bundles over X.

Proof. One can carry through the standard proof (cf. [MS]). The only point to establish is that a Real bundle is locally trivial in the category of Real bundles. This is shown for example in [A]. □

It follows that the limiting Real space $G^\infty \cong G^\infty(\mathbb{C}^\infty \oplus \mathbb{C}^\infty) = \mathbf{BU}$ classifies Atiyah's KR-theory ([A]), and hence this equivariant infinite loop space structure on $\mathbf{BU}$ gives an equivariant spectrum $\mathfrak{KR}$ whose associated $RO(\mathbb{Z}_2)$-graded cohomology theory is an enhancement of KR-theory.

In what follows, we show how to construct another $\mathbb{Z}_2\mathcal{I}_*$-functor using constructions with algebraic cycles. The resulting equivariant infinite loop space will then be used to provide *characteristic classes* for the $RO(\mathbb{Z}_2)$-graded KR-theory.

Example 6.8. (The algebraic cycle functor) Consider the functor defined by setting $T_Z(V) = \mathcal{Z}^v(\mathbb{P}(V_{\mathbb{C}} \oplus V_{\mathbb{C}}))$, the topological group of codimension-v cycles in $\mathbb{P}(V_{\mathbb{C}} \oplus V_{\mathbb{C}})$, with $1_Z = 1_G$. To a morphism $f : V \to W$ we associate

$$T_Z(f)c \;=\; \mathbb{P}(f_{\mathbb{C}}(V_{\mathbb{C}})^\perp \oplus \{0\}) \,\#\, (f_{\mathbb{C}} \oplus f_{\mathbb{C}})_*(c),$$

and we define ω_Z by

$$\omega_Z(c, c') = \tau_*(c\#c').$$

Using the same arguments as in [LLM$_1$], it can be shown that (T_Z, ω_Z) is a $\mathbb{Z}_2\mathcal{I}_*$- functor with the $\mathbb{Z}_2$-action given by conjugation, and hence

$$T_Z(\mathbb{C}^\infty) \;=\; \mathcal{Z}^\infty_{\mathbb{C}}$$

is an equivariant $\mathcal{L}$-space. In fact it is an equivariant E_∞-ring space which is additively $\mathbb{Z}_2$-group complete and therefore it is equivalent to the 0^{th} space of an equivariant E_∞-ring spectrum, which we denote by $\mathfrak{Z}_{\mathbb{C}}$.

The join operation ω_Z has various properties which yield important results:

(1) The join is multiplicative with respect to degree of cycles, in other words

$$\deg \omega_Z(c, c') = \deg c \cdot \deg c';$$

(2) If c is an averaged cycle then, for any fixed cycle c', the join $\omega_Z(c, c')$ is also an averaged cycle. In other words, the averaged cycles form an "ideal" within the fixed cycles.

Now, let $\mathcal{Z}_{\mathbb{C}}^\infty(1) \subset \mathcal{Z}_{\mathbb{C}}^\infty$ be the subspace consisting of the cycles of degree one. Since the join operation ω_Z is equivariant and multiplicative on degrees, one concludes that:

a. $\mathcal{Z}_{\mathbb{C}}^\infty(1)$ is also an equivariant $\mathbb{Z}_2$-group-complete $\mathcal{L}$-space, since it is connected and its fixed point set is connected. It then follows that $\mathcal{Z}_{\mathbb{C}}^\infty(1)$ carries a structure of an equivariant infinite loop space of its own, and hence it is equivalent to the 0^{th} space of another equivariant spectrum, which we denote by $\mathfrak{M}_{\mathbb{R}}$ to emphasize the fact that we are equivariant delooping the *multiplicative units* of $\mathcal{Z}_{\mathbb{C}}^\infty$.

b. Given $V \in \mathbb{Z}_2\mathcal{I}_*$, the "forgetful map"

$$\Phi_V : \mathcal{Z}_{\mathbb{C}}^v(\mathbb{P}(V_{\mathbb{C}} \oplus V_{\mathbb{C}})) \longrightarrow \mathcal{Z}_{\mathbb{R}}^{2v}(\mathbb{P}(V_{\mathbb{C}} \oplus \bar{V}_{\mathbb{C}} \oplus V_{\mathbb{C}} \oplus \bar{V}_{\mathbb{C}})),$$

which sends c to $\omega_Z(c \oplus c)$, is not a group homomorphism. Nevertheless, the preservation of degrees by the join implies that the maps Φ_V define a map of (non-equivariant) $\mathcal{I}_*$-functors between $\mathcal{Z}_{\mathbb{C}}^*$ and $\mathcal{Z}_{\mathbb{R}}^*$, which preserves cycles of degree 1. In particular, they induce a map of $\mathcal{L}$-spaces $\Phi : \mathcal{Z}_{\mathbb{C}}^\infty(1) \to \mathcal{Z}_{\mathbb{R}}^\infty(1)$.

All of this discussion, in fact all the discussion in sections 4, 5, and 6 of [BLLMM] and section 3 of [LLM$_1$], which include material on Chow monoid functors, carries over directly to our spaces of algebraic cycles.

Theorem 6.9.

(1) *The limiting topological group $\mathcal{Z}_{\mathbb{C}}^\infty$ is an equivariant E_∞-ring space which forms the 0-level space of an equivariant E_∞-ring spectrum $\mathfrak{Z}_{\mathbb{C}}$. The fixed point set $\mathcal{Z}_{\mathbb{R}}^\infty$ is a (non-equivariant) E_∞-ring space which forms the 0-level space of an E_∞-ring spectrum $\mathfrak{Z}_{\mathbb{R}}$. The inclusion $\Gamma : \mathcal{Z}_{\mathbb{R}}^\infty \hookrightarrow \mathcal{Z}_{\mathbb{C}}^\infty$ extends to a map of (non-equivariant) ring spectra $\Gamma : \mathfrak{Z}_{\mathbb{R}} \to \mathfrak{Z}_{\mathbb{C}}$.*

(2) *The quotient group $\tilde{\mathcal{Z}}_{\mathbb{R}}^\infty \overset{\text{def}}{=} \mathcal{Z}_{\mathbb{R}}^\infty/(\mathcal{Z}_{\mathbb{C}}^\infty)^{av}$ is also an E_∞-ring space, and the quotient map $\rho : \mathcal{Z}_{\mathbb{R}}^\infty \to \tilde{\mathcal{Z}}_{\mathbb{R}}^\infty$ is a map of E_∞-ring spaces. Hence, $\tilde{\mathcal{Z}}_{\mathbb{R}}^\infty$ is the 0^{th} space of an E_∞-ring spectrum $\tilde{\mathfrak{Z}}_{\mathbb{R}}$, and there is a natural map of spectra $\rho : \mathfrak{Z}_{\mathbb{R}} \to \tilde{\mathfrak{Z}}_{\mathbb{R}}$. Similarly, $\tilde{\mathcal{Z}}_{\mathbb{R}}^\infty(1)$ is an infinite loop space under the operation induced by the join, which makes it into the 0^{th} space of an spectrum $\tilde{\mathfrak{M}}_{\mathbb{R}}$.*

(3) *$\mathcal{Z}_{\mathbb{C}}^\infty(1)$ carries an infinite loop space structure which enhances the algebraic join, and makes it into the 0-level space of an equivariant spectrum $\mathfrak{Z}_{\mathbb{C}}$. The fixed point set $\mathcal{Z}_{\mathbb{R}}^\infty$ is a (non-equivariant) E_∞-ring space which forms the 0-level space of a*

spectrum $\mathfrak{M}_{\mathbb{R}}$. The inclusion $\Gamma : \mathcal{Z}^\infty_{\mathbb{R}}(1) \hookrightarrow \mathcal{Z}^\infty_{\mathbb{C}}(1)$ extends to a map of (non-equivariant) spectra $\mathfrak{M}_{\mathbb{R}} \to \mathfrak{M}_{\mathbb{C}}$.

(4) *The canonical "forgetful map" $\Phi : \mathcal{Z}^\infty_{\mathbb{C}}(1) \to \mathcal{Z}^\infty_{\mathbb{R}}(1)$ induces a map of (non-equivariant) spectra $\Phi : \mathfrak{M}_{\mathbb{C}} \to \mathfrak{M}_{\mathbb{R}}$.*

An important feature of $\mathcal{Z}^\infty_{\mathbb{C}}(1)$ comes from the fact that the inclusion

$$G^v(\mathbb{P}(V_{\mathbb{C}} \oplus V_{\mathbb{C}})) \subset \mathcal{Z}^v(\mathbb{P}(V_{\mathbb{C}} \oplus V_{\mathbb{C}}))(1),$$

as effective cycles of degree 1, is a natural transformation of $\mathbb{Z}_2\mathcal{I}_*$-functors, and the resulting map

$$\mathbf{BU} \to \mathcal{Z}^\infty_{\mathbb{C}}(1)$$

is an equivariant infinite loop space map. This fact, together with the discussion above and [BLLMM], gives the following result.

Theorem 6.10.

(1) *The canonical equivariant inclusion $\mathbf{BU} \to \mathcal{Z}^\infty_{\mathbb{C}}(1)$ extends to a morphism*

$$\widetilde{\mathbf{c}} : \mathfrak{K}\mathfrak{R} \to \mathfrak{M}_{\mathbb{C}}$$

of $\mathbb{Z}_2$-equivariant spectra. Passing to fixed point sets gives maps of (non-equivariant) spectra $P : \mathfrak{K}\mathfrak{o} \to \mathfrak{M}_{\mathbb{R}}$ from connective KO-theory to $\mathfrak{M}_{\mathbb{R}}$, and $c : \mathfrak{K}\mathfrak{u} \to \mathfrak{M}_{\mathbb{C}}$ from connective K-theory to $\mathfrak{M}_{\mathbb{C}}$. These maps fit into a commutative diagram of spectra

$$\begin{array}{ccc} \mathfrak{K}\mathfrak{o} & \xrightarrow{\gamma} & \mathfrak{K}\mathfrak{u} \\ {\scriptstyle \widetilde{\mathbf{c}}}\downarrow & & \downarrow{\scriptstyle c} \\ \mathfrak{M}_{\mathbb{R}} & \xrightarrow[\Gamma]{} & \mathfrak{M}_{\mathbb{C}} \end{array}$$

which extends the commutative diagram

$$\begin{array}{ccc} \mathbf{BO} & \xrightarrow{\gamma} & \mathbf{BU} \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle c} \\ \mathcal{Z}^\infty_{\mathbb{R}} & \xrightarrow[\Gamma]{} & \mathcal{Z}^\infty_{\mathbb{C}}, \end{array}$$

where the map $c : BU \to \mathcal{Z}^\infty_{\mathbb{C}}$ classifies the total Chern class. *The composition $\mathfrak{K}\mathfrak{o} \to \mathfrak{M}_{\mathbb{R}} \to \tilde{\mathfrak{M}}_{\mathbb{R}}$ is an extension to spectra level of the classifying map $\mathbf{BO} \to \mathcal{Z}^\infty_{\mathbb{R}}(1) \to \tilde{\mathcal{Z}}^\infty_{\mathbb{R}}(1)$ for the total Stiefel-Whitney class.*

Analogous results for the groups of quaternionic cycles will be established in the companion paper [LLM_3].

A natural question now arises: *What is the equivariant cohomology theory classified by the $\mathbb{Z}_2$-spectrum $\mathfrak{Z}_{\mathbb{C}}$?* In his thesis Pedro dos Santos has established the following beautiful results. To state them we briefly recall some concepts from equivariant homotopy theory (cf. [M$_4$].)

Let G be a finite group and $\underline{M}$ a Mackey functor for G. To each real representation V of G there is an Eilenberg-MacLane space $K(\underline{M}, V)$ which classifies the ordinary equivariant cohomology group $H^V_G(\bullet; \underline{M})$ in dimension V with coefficients in the Mackey functor $\underline{M}$. These fit together to give an equivariant spectrum $\mathbb{K}(\underline{M}, 0)$ which classifies the full RO_G-graded equivariant cohomology with coefficients in $\underline{M}$.

We now specialize to the group $G = \mathbb{Z}_2$ and $\underline{M} = \underline{\mathbb{Z}}$, the Mackey functor constant at $\mathbb{Z}$. For each n we consider the fundamental respresentation $\mathbb{R}^{n,n} = \mathbb{C}^n$ of $\mathbb{Z}_2$ given by complex conjugation.

Theorem 6.11. **(dos Santos [dS])** *There is a canonical $\mathbb{Z}_2$-equivariant homotopy equivalence*

$$\mathcal{Z}^\infty_{\mathbb{C}} \;\cong\; \prod_{n=0}^{\infty} K(\underline{\mathbb{Z}}, \mathbb{R}^{n,n}). \tag{6.1}$$

This extends to an equivalence of $\mathbb{Z}_2$-equivariant ring spectra

$$\mathfrak{Z}_{\mathbb{C}} \;\cong\; \mathbb{K}(\underline{\mathbb{Z}}, 0) \times \mathbb{K}(\underline{\mathbb{Z}}, \mathbb{R}^{1,1}) \times \mathbb{K}(\underline{\mathbb{Z}}, \mathbb{R}^{2,2}) \times \dots$$

where $\mathbb{K}(\underline{\mathbb{Z}}, 0)$ is the equivariant Eilenberg- MacLane spectrum and $\mathbb{K}(\underline{\mathbb{Z}}, \mathbb{R}^{n,n})$ is the connective equivariant spectrum with $\Omega^{\mathbb{R}^{n,n}}\mathbb{K}(\underline{\mathbb{Z}}, \mathbb{R}^{n,n}) \cong \mathbb{K}(\underline{\mathbb{Z}}, 0)$, and where the ring structure is given by the equivariant cup product pairing.

For a $\mathbb{Z}_2$-space X we denote by $H^*_{\mathbb{Z}_2}(X; \underline{\mathbb{Z}})$ the full $RO_{\mathbb{Z}_2}$-graded equivariant cohomology ring of X with coefficients in the Mackey functor $\underline{\mathbb{Z}}$. We abbreviate $H^{\mathbb{R}^{n,n}}_{\mathbb{Z}_2}(X; \underline{\mathbb{Z}}) \equiv H^{n,n}_{\mathbb{Z}_2}(X; \underline{\mathbb{Z}})$

Theorem 6.12. **(Dugger and dos Santos)** *There is a canonical ring homomorphism*

$$H^*_{\mathbb{Z}_2}(\mathbf{BU}; \underline{\mathbb{Z}}) \cong R[\widetilde{c}_1, \widetilde{c}_2, \widetilde{c}_3, \dots]$$

where

$$\widetilde{c}_n \in H^{n,n}_{\mathbb{Z}_2}(\mathbf{BU}; \underline{\mathbb{Z}})$$

for each n and $R = H^*_{\mathbb{Z}_2}(\mathbf{pt}; \underline{\mathbb{Z}})$ is the coefficient ring.

Furthermore, let $\widetilde{\iota}_{n,n}$ denote the fundamental class of $K(\underline{\mathbb{Z}}, \mathbb{R}^{n,n})$. Then with respect to the splitting (6.1) the natural $\mathbb{Z}_2$-map

$$P : \mathbf{BU} \;\longrightarrow\; \mathcal{Z}^\infty_{\mathbb{C}}$$

satisfies

$$P^*\left(\widetilde{\iota}_{n,n}\right) = \widetilde{c}_n$$

Note . The first assertion of Theorem 6.12 is due to Dan Dugger [D] and the second to dos Santos [dS].

Theorem 6.12 shows that the inclusion map $\widetilde{c} : \mathbf{BU} \to \mathcal{Z}^\infty_{\mathbb{C}}(1)$ naturally classifies the *total Chern class in (full $RO_{\mathbb{Z}_2}$-graded) equivariant cohomology.* Thus for Real spaces X, $\widetilde{c}$ determines a natural transformation

$$\widetilde{c} : KR(X) \longrightarrow \bigoplus_{n\geq 0} H^{n,n}_{\mathbb{Z}_2}(X; \underline{\mathbb{Z}})$$

and the property that $\widetilde{c}(V \oplus V') = \widetilde{c}(V)\#\widetilde{c}(V')$, together with Theorem 6.12, shows that

$$\widetilde{c}(E \oplus E') = \widetilde{c}(E) \cup \widetilde{c}(E')$$

for all $E, E' \in KR(X)$. Theorem 6.10 shows that the equivariant infinite loop structure on $\mathcal{Z}^\infty_{\mathbb{C}}(1)$ corresponding to the spectrum $\mathfrak{M}_{\mathbb{C}}$ makes this total Chern class map $\widetilde{c}$ *an equivariant infinite loop map.* This is the full $\mathbb{Z}_2$-equivariant version of Segal's conjecture settled in [BLLMM].

In fact under the forgetful functor the class $\widetilde{c}$ becomes the ordinary total Chern class c, and the map $\widetilde{\mathbf{c}} : \mathfrak{K}\mathfrak{R} \to \mathfrak{M}_{\mathbb{C}}$ of equivariant spectra becomes the map $\mathbf{c} : \mathfrak{K}\mathfrak{u} \to \mathfrak{M}_{\mathbb{C}}$ of non-equivariant spectra studied in [BLLMM].

More interesting perhaps is the restriction of $\widetilde{c}$ to the fixed-point set $\mathbf{BO} \subset \mathbf{BU}$. This gives a characteristic class for real bundles which is a mixture of $\mathbb{Z}$ and $\mathbb{Z}_2$ classes and satisfies Whitney duality. We examine this next.

§7. A new total characteristic class.

The mappings

$$P : G^q_{\mathbb{R}}(\mathbb{P}(V)) \longrightarrow \mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}(V))$$

which stabilize to

$$P : BO_q \longrightarrow \mathcal{Z}^q_{\mathbb{R}} \qquad \text{and} \qquad P : \mathbf{BO} \longrightarrow \mathcal{Z}^\infty_{\mathbb{R}}$$

represent a "total" characteristic class which is in complete analogy with the *total Chern class*:

$$c : G^q_{\mathbb{C}}(\mathbb{P}(V)) \longrightarrow \mathcal{Z}^q_{\mathbb{C}}(\mathbb{P}(V))$$

$$c : BU_q \longrightarrow \mathcal{Z}^q_{\mathbb{R}} \qquad \text{and} \qquad c : \mathbf{BU} \longrightarrow \mathcal{Z}^\infty_{\mathbb{R}}$$

and the *total Stiefel-Whitney class*:

$$w : G^q_{\mathbb{R}}(\mathbb{P}(V)) \longrightarrow \widetilde{\mathcal{Z}}^q_{\mathbb{R}}(\mathbb{P}(V))$$

$$w : BO_q \longrightarrow \widetilde{\mathcal{Z}}^q_{\mathbb{R}} \qquad \text{and} \qquad w : \mathbf{BO} \longrightarrow \widetilde{\mathcal{Z}}^\infty_{\mathbb{R}}.$$

This new class has the property that for real vector bundles E and F over a space X,

$$P(E \oplus F) = P(E) \cup P(F). \tag{7.1}$$

As a map from **BO** to $\mathcal{Z}_{\mathbb{R}}^{\infty}$ it is an infinite loop map. It fits into a pattern of infinite loop diagrams as we saw in §6. This compelling picture makes the further study of P intriguing.

Recall that we have a commutative diagram:

$$\begin{array}{ccccc} & & & & \mathcal{Z}_{\mathbb{C}}^{\infty} \\ & & & \nearrow & \\ \mathbf{BO} & \xrightarrow{P} & \mathcal{Z}_{\mathbb{R}}^{\infty} & & \\ & & & \searrow & \\ & & & & \widetilde{\mathcal{Z}}_{\mathbb{R}}^{\infty} \end{array}$$

where the upper map represents the total Chern class of the complexification (essentially the total Pontrjagin class) and the lower map is the total Steifel-Whitney class. However, P contains much more information. From the splitting in Theorem 3.3 the map P is seen to represent a certain sum of integral and mod 2 cohomology classes. Thus P is a particular arrangement of Pontryagin and Stiefel-Whitney classes. Exactly which arrangement has been determined by dos Santos.

Theorem 7.1. ([dS]) *For $k < n$, one has*

$$P^*(\iota_{n,k}) = Sq^k w_n$$

where Sq^k denotes the k^{th} Steenrod operation and w_n denotes the n^{th} Stiefel-Whitney class.

When $k = n$ and n is even, $P^*(\iota_{n,n})$ is the n^{th} Pontrjagin class p_n.

There is a second construction that one can associate to Real bundles.

Construction 7.2 To a Real map $f : X \to G^q(\mathbb{P}(\mathbb{C}^n))$ classifying a Real bundle $E_f \to X$, we associate the mapping

$$\widetilde{f} : X/\mathbb{Z}_2 \longrightarrow \mathcal{Z}^q(\mathbb{P}(\mathbb{C}^N))^{av}$$

defined by $\widetilde{f}([x]) = f(x) + f(\rho x) = f(x) + \rho f(x)$. Let $\iota_{n,k}$ denote the fundamental class of the factor $K(I_{n,k}, n+k)$ in the canonical splitting of $\mathcal{Z}_{\mathbb{R}}^q$ given in Theorem 3.3. Then

$$\widetilde{f}^*(\iota_{n,k}) \in H^*(X/\mathbb{Z}_2)$$

is an invariant of the Real bundle E_f.

Example 7.3. Consider the commutative diagram of Real spaces

$$\begin{array}{ccc} \mathbb{P}^n & \longrightarrow & G^n(\mathbb{P}(\mathbb{C}^N)) \\ \downarrow & & \downarrow \\ \mathcal{Z}_0(\mathbb{P}^n) & \longrightarrow & \mathcal{Z}^n(\mathbb{P}(\mathbb{C}^N)) \end{array}$$

where the left vertical map is the standard inclusion, the horizontal maps are complex suspension, and the right vertical map is our $\mathbb{Z}_2$- characteristic map. Let $\mathbb{P}^{n,k} \subset \mathbb{P}^n$ be the Real spaces which are introduced in the proof of Theorem 9.1. Composing with suspension gives a Real mapping $\mathbb{P}^{n,k} \longrightarrow G^n(\mathbb{P}(\mathbb{C}^N))$, for which 7.1 yields a map $\widetilde{f}$: $\mathbb{P}^{n,k}/\mathbb{Z}_2 \longrightarrow \mathcal{Z}^q(\mathbb{P}(\mathbb{C}^N))^{av}$. It follows from the discussion (9.11) ff. that the corresponding classes $\widetilde{f}^*(\iota_{n,k})$ are non-trivial for $0 < k \leq n$.

Note 7.4. Let $\widetilde{P}(E)$ denote the total class of a Real bundle E constructed in 7.2. Then we have

$$\widetilde{P}(E \oplus E') \;=\; \widetilde{P(E)\#P(E')}$$

This class satisfies the addition relation:

$$\widetilde{P}(E \oplus E') + \widetilde{P}(E \oplus \overline{E}') \;=\; \widetilde{P}(E)\#\widetilde{P}(E')$$

§8. Proof of Theorems 3.3 and 3.4

Consider the Real vector space $(\mathbb{C}^{n+1}, \tau)$ where τ is complex conjugation, and denote the corresponding projective space by $\mathbb{P}^n_{\mathbb{C}}$ and its real form (the τ-fixed-point set) by $\mathbb{P}^n_{\mathbb{R}}$. Our first step reduces everything to the case of 0-cycles.

Proposition 8.1. *Fix integers $q \leq N$. Then*

a. *Iterations of the algebraic suspension give canonical homotopy equivalences:*

$$\mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}^N_{\mathbb{C}}) \cong \mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}^q_{\mathbb{C}}) = \mathcal{Z}_{0,\mathbb{R}}(\mathbb{P}^q_{\mathbb{C}}),$$
$$\mathcal{Z}^q(\mathbb{P}^N_{\mathbb{C}})^{av} \cong \mathcal{Z}^q(\mathbb{P}^q_{\mathbb{C}})^{av} = \mathcal{Z}_0(\mathbb{P}^q_{\mathbb{C}})^{av},$$
$$\widetilde{\mathcal{Z}}^q_{\mathbb{R}}(\mathbb{P}^N_{\mathbb{C}}) \cong \widetilde{\mathcal{Z}}^q_{\mathbb{R}}(\mathbb{P}^q_{\mathbb{C}}) = \widetilde{\mathcal{Z}}_{0,\mathbb{R}}(\mathbb{P}^q_{\mathbb{C}}).$$

b. *The short exact sequence of topological abelian groups:*

$$0 \to \mathcal{Z}^q(\mathbb{P}^q_{\mathbb{C}})^{av} \to \mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}^q_{\mathbb{C}}) \xrightarrow{p} \widetilde{\mathcal{Z}}^q_{\mathbb{R}}(\mathbb{P}^q_{\mathbb{C}}) \to 0.$$

is a principal fibration.

Proof. Part **a** follows from repeated applications of Theorem 3.2. To prove part **b**, consider the monoid $C \stackrel{\text{def}}{=} \amalg_{d\geq 0} SP_d(\mathbb{P}^q_{\mathbb{C}})^{fix}$ and its closed submonoid $C' \stackrel{\text{def}}{=} \amalg_{d'\geq 0} SP_{2d'}(\mathbb{P}^q_{\mathbb{C}})^{av}$. Note that the naïve group completions of C and C' are $\mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}^q_{\mathbb{C}})$ and $\mathcal{Z}^q(\mathbb{P}^q_{\mathbb{C}})^{av}$, respectively. Since complex conjugation induces a real analytic map on all products $SP_d(\mathbb{P}^q_{\mathbb{C}})\times SP_{d'}(\mathbb{P}^q_{\mathbb{C}})$, preserving filtrations by degrees, one can provide equivariant triangulations to all such products, making (C, C') into a triangulated pair. It follows that (C, C') satisfies the hypothesis of [Li$_2$, Theorem 5.2], which then implies the desired result. □

The next result makes thorough use of the identification of $\mathbb{P}^n_{\mathbb{C}}$ with the n-fold symmetric product $\mathrm{S}P^n(\mathbb{P}^1_{\mathbb{C}})$ of $\mathbb{P}^1_{\mathbb{C}}$, and the fact that the complex conjugation involution on $\mathbb{P}^n_{\mathbb{C}}$ is induced by the complex conjugation τ on $\mathbb{P}^1_{\mathbb{C}}$ under this identification. We are grateful to the referee for a very nice technical improvement of our original version of this Proposition which was based on a construction in [FL].

For any compact space Y let $\mathcal{Z}_0(Y)_o$ denote the connected component of 0 in $\mathcal{Z}_0(Y)$.

Proposition 8.2. *There exist canonical equivariant homeomorphisms*

$$\mathcal{Z}^q(\mathbb{P}^q_{\mathbb{C}}) \;\cong\; \prod_{n=0}^{q}\{\mathcal{Z}_0(\mathbb{P}^n_{\mathbb{C}})/\mathcal{Z}_0(\mathbb{P}^{n-1}_{\mathbb{C}})\} \;\cong\; \mathbb{Z}\times\prod_{n=1}^{q}\mathcal{Z}_0(S^{2n})_o \tag{8.1}$$

where

$$S^{2n} \;=\; \mathbb{C}^n\cup\{\infty\} \;=\; \mathbb{P}^n_{\mathbb{C}}/\mathbb{P}^{n-1}_{\mathbb{C}} \tag{8.2}$$

with $\mathbb{Z}_2$-action given by complex conjugation.

Proof. Choose a basepoint $\infty\in\mathbb{P}^1_{\mathbb{C}}$ which is fixed under the complex conjugation τ, i.e. a real point. Then, for each $n\le q$, the canonical inclusion

$$\begin{array}{ccc} \mathbb{P}^n_{\mathbb{C}} = \mathrm{S}P^n(\mathbb{P}^1_{\mathbb{C}}) & \longrightarrow & \mathbb{P}^q_{\mathbb{C}} = \mathrm{S}P^q(\mathbb{P}^1_{\mathbb{C}}) \\ \sigma & \longmapsto & \sigma+(q-n)\cdot\infty \end{array} \tag{8.3}$$

is $\mathbb{Z}_2$-equivariant and induces an injective equivariant homomorphism

$$\mathcal{Z}_0(\mathbb{P}^n_{\mathbb{C}})\to\mathcal{Z}_0(\mathbb{P}^q_{\mathbb{C}}). \tag{8.4}$$

We now define a continuous equivariant homomorphism $\rho_q:\mathcal{Z}_0(\mathbb{P}^q_{\mathbb{C}})\to\mathcal{Z}_0(\mathbb{P}^{q-1}_{\mathbb{C}})$ by setting

$$\rho_q(x_1+\cdots+x_q) \;=\; \sum_{\substack{I\subset\{1,\ldots,q\}\\ \neq}}(-1)^{q-1-|I|}(x_I+(q-1-|I|)\cdot\infty) \tag{8.5}$$

for points $x_1+\cdots+x_q\in SP^q(\mathbb{P}^1_{\mathbb{C}})=\mathbb{P}^q_{\mathbb{C}}$ and extending linearly. Here $x_I\equiv x_{i_1}+x_{i_2}+\cdots+x_{i_k}$ with $k=|I|$, and as always we take $x_I+(q-1-|I|)\cdot\infty\in SP^{q-1}(\mathbb{P}^1_{\mathbb{C}})=\mathbb{P}^{q-1}_{\mathbb{C}}$ It is straightforward to verify that this mapping is a **continuous retraction** onto the subgroup given in (8.4) with $n=q-1$. Thus we obtain an equivariant direct product decomposition of topological abelian groups

$$\mathcal{Z}_0(\mathbb{P}^q_{\mathbb{C}}) \;=\; \mathcal{Z}_0(\mathbb{P}^{q-1}_{\mathbb{C}})\times(\mathcal{Z}_0(\mathbb{P}^q_{\mathbb{C}})/\mathcal{Z}_0(\mathbb{P}^{q-1}_{\mathbb{C}}))$$

which under iteration yields a canonical equivariant direct product decomposition

$$\mathcal{Z}_0(\mathbb{P}^q_{\mathbb{C}}) \;=\; \prod_{n=o}^{q}\mathcal{Z}_0(\mathbb{P}^n_{\mathbb{C}})/\mathcal{Z}_0(\mathbb{P}^{n-1}_{\mathbb{C}}).$$

Now note that $\mathcal{Z}_0(\mathbb{P}^0_{\mathbb{C}})=\mathbb{Z}$ and $\mathcal{Z}_0(\mathbb{P}^n_{\mathbb{C}})/\mathcal{Z}_0(\mathbb{P}^{n-1}_{\mathbb{C}})=\mathcal{Z}_0(\mathbb{P}^n_{\mathbb{C}}/\mathbb{P}^{n-1}_{\mathbb{C}})_o$ for $n>0$. ☐

Applying the averaging, fixed-point and quotient functors gives similar (non-equivariant) splittings for our real cycle spaces.

Corollary 8.3. *There exist canonical direct product decompositions of topological abelian groups:*

$$(8.6)\qquad \mathcal{Z}_0(\mathbb{P}^q_{\mathbb{C}})^{av} \cong 2\mathbb{Z} \times \prod_{n=1}^{q} \mathcal{Z}_0(S^{2n})^{av}_o, \qquad \mathcal{Z}_{0,\mathbb{R}}(\mathbb{P}^q_{\mathbb{C}}) \cong \mathbb{Z} \times \prod_{n=1}^{q} \mathcal{Z}_0(S^{2n})^{fix}_o$$

*where projection onto the first factor is the degree. These splittings are compatible, and the fibration in 8.1(**b**) splits as a product $p = \prod_n p_n$ of principal fibrations:*

$$(8.7)\qquad 0 \longrightarrow \mathcal{Z}_0(S^{2n})^{av}_o \longrightarrow \mathcal{Z}_0(S^{2n})^{fix}_o \xrightarrow{p_n} \mathcal{Z}_0(S^n)_o \otimes \mathbb{Z}_2 \longrightarrow 0$$

for $n > 0$ and the projection $p_0 : \mathbb{Z} \to \mathbb{Z}_2$. In particular we obtain a canonical direct product decomposition

$$(8.8)\qquad \widetilde{\mathcal{Z}}_{0,\mathbb{R}}(\mathbb{P}^q_{\mathbb{C}}) \cong \prod_{n=0}^{q} \mathcal{Z}_0(S^n)_o \otimes \mathbb{Z}_2 \equiv \prod_{n=0}^{q} K(\mathbb{Z}_2, n).$$

Proof. The first assertion and the compatibility of the splittings follow straightforwardly from 8.2. Notice that one has canonical topological isomorphisms

$$\mathcal{Z}_0(S^{2n})^{fix}_o / \mathcal{Z}_0(S^{2n})^{av}_o \cong \mathcal{Z}_0(\{S^{2n}\}^{fix})_o \otimes \mathbb{Z}_2 \cong \mathcal{Z}_0(S^n)_o \otimes \mathbb{Z}_2$$

for $n \geq 1$ and that $\mathcal{Z}_0(S^n)_o \otimes \mathbb{Z}_2 = K(\mathbb{Z}_2, n)$ by the Dold-Thom Theorem [DT]. This establishes (8.8). The fact that the maps p_n are principal fibrations follows from [Li$_2$, Theorem 5.2] as in the proof of 8.1. $\square$

The next technical result is clearly needed for our subsequent computations.

Lemma 8.4. *For any compact space X with a $\mathbb{Z}_2$-action there is a natural degree-preserving topological isomorphism: $\mathcal{Z}_0(X)^{av} \cong \mathcal{Z}_0(X/\mathbb{Z}_2)$, and therefore by the Dold-Thom Theorem*

$$\pi_k \mathcal{Z}_0(X)^{av} = H_k(X/\mathbb{Z}_2; \mathbb{Z}) \qquad \text{for all } k.$$

Proof. Consider the topological homomorphism $\psi : \mathcal{Z}_0(X) \to \mathcal{Z}_0(X)^{av} \subset \mathcal{Z}_0(X)$ defined by $\psi(\sigma) = \sigma + \tau * \sigma$, where $\tau * \sigma$ denotes the action of the generator of $\mathbb{Z}_2$ on σ. Since X is compact, it follows from the description of the topology of $\mathcal{Z}_0(X)$ that ψ is a closed map, which clearly surjects onto $\mathcal{Z}_0(X)^{av}$. The composition $X \to \mathcal{Z}_0(X) \to \mathcal{Z}_0(X)^{av}$ clearly factors through the projection $\pi : X \to X/\mathbb{Z}_2$, and hence the universal property of the functor $\mathcal{Z}_0(-)$ gives a continuous homomorphism $\Psi : \mathcal{Z}_0(X/\mathbb{Z}_2) \to \mathcal{Z}_0(X)^{av}$ such that $\Psi \circ \pi_* = \psi$, where $\pi_* : \mathcal{Z}_0(X) \to \mathcal{Z}_0(X/\mathbb{Z}_2)$ is the projection induced by π. It is a routine verification to see that Ψ is injective, and hence a closed continuous bijection. $\square$

Proposition 8.5. *For each $n > 0$ there exists a canonical cross-section of the principal fibration (8.7). Therefore there exist canonical splittings:*

$$\mathcal{Z}_0(S^{2n})^{fix}_o \cong \mathcal{Z}_0(S^{2n})^{av}_o \times \widetilde{\mathcal{Z}}_{0,\mathbb{R}}(S^{2n})$$

for $n > 0$, and so also a canonical splitting:

$$(8.9)\qquad \mathcal{Z}_{0,\mathbb{R}}(\mathbb{P}^q_{\mathbb{C}})_o \cong \mathcal{Z}_0(\mathbb{P}^q_{\mathbb{C}})^{av}_o \times \widetilde{\mathcal{Z}}_{0,\mathbb{R}}(\mathbb{P}^q_{\mathbb{C}})$$

Proof. We want to produce a canonical homotopy section for the fibration $\mathcal{Z}_0(S^{2n})_o^{av} \to \mathcal{Z}_0(S^{2n})_o^{fix} \to \mathcal{Z}_0(S^n)_o \otimes \mathbb{Z}_2$ where S^{2n} is the one-point compactification of $\mathbb{C}^n$ and $S^n \subset S^{2n}$ is the sphere of real points. Define

$$\epsilon : S^n \to \mathcal{Z}_0(S^{2n})_o^{fix}$$

by $\epsilon(x) = x - x_\infty$, where x_∞ is the base point, and let ϵ_* be its extension to $\mathcal{Z}_0(S^n)$ Then there is a commutative diagram

$$\begin{array}{ccc} \mathcal{Z}_0(S^n)_o & \xrightarrow{\epsilon_*} & \mathcal{Z}_0(S^{2n})_o^{fix} \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle P} \\ \mathcal{Z}_0(S^n)_o \otimes \mathbb{Z}_2 & = & \mathcal{Z}_0(S^{2n})_o^{fix}/\mathcal{Z}_0(S^{2n})_o^{av}, \end{array} \tag{8.10}$$

where Q and P denote the quotient maps.

Let $f_2 : S^n \to S^n$ be the map of degree 2, fixed in Appendix A, §A.1. It follows from standard properties of H-spaces that $\epsilon \circ f_2$ is homotopic to 2ϵ, and it is clear that the map 2ϵ factors through the averaged cycles $\mathcal{Z}_0(S^{2n})_o^{av}$. On the other hand, it is easy to see that one has homeomorphisms

$$\{\mathbb{P}^n_{\mathbb{C}}/\mathbb{P}^{n-1}_{\mathbb{C}}\}/\mathbb{Z}_2 \cong S^{2n}/\mathbb{Z}_2 \cong S^n \# \mathbb{P}^{n-1}_{\mathbb{R}},$$

where $\#$ denotes the real join of topological spaces. Combining this fact with Lemma 8.4, one obtains that $\mathcal{Z}_0(S^{2n})_o^{av} = \mathcal{Z}_0(\mathbb{P}^n_{\mathbb{C}}/\mathbb{P}^{n-1}_{\mathbb{C}})_o^{av} \cong \mathcal{Z}_0(\{\mathbb{P}^n_{\mathbb{C}}/\mathbb{P}^{n-1}_{\mathbb{C}}\}/\mathbb{Z}_2)_o$, and the latter space is $(n+1)$-connected. Therefore, 2ϵ is homotopic to zero.

Now, Lemma A.1 and Corollary A.4 provide a canonical map

$$H : \mathcal{Z}_0(S^n)_o \otimes \mathbb{Z}_2 \to \mathcal{Z}_0(S^{2n})_o^{fix},$$

unique up to homotopy, with the property that

$$H \circ Q \simeq \epsilon_*. \tag{8.11}$$

Let us apply Corollary A.4 once again, with $Y = \mathcal{Z}_0(S^n)_o \otimes \mathbb{Z}_2$ and h being the composition $S^n \xrightarrow{j} \mathcal{Z}_0(S^n)_o \xrightarrow{Q} \mathcal{Z}_0(S^n)_o \otimes \mathbb{Z}_2$ where $j(x) = x - x_\infty$. Using (8.11), we obtain $(P \circ H) \circ Q \simeq P \circ \epsilon_*$, and since $P \circ \epsilon_* = Q$, cf. (8.10), we conclude that

$$(P \circ H) \circ Q \simeq Q = id \circ Q = (Q \circ j)_*.$$

Therefore both $(P \circ H)$ and id satisfy the same condition in Corollary A.4, and since $\pi_{n+1}(\mathcal{Z}_0(S^n) \otimes \mathbb{Z}_2) = 0$ one obtains $P \circ H \simeq id$. Therefore, H is the desired homotopy section of P. This proves the first assertion. The second follows easily from 8.3. ☐

There are two distinct ways to complete the computation at this point. The first one occurs at the space level and the other, suggested by the referee, takes place at the level of simplicial groups. The first is elementary and involves the Dold-Thom Theorem [DT], while the second has a more "motivic" nature. We shall present them both.

First, apply Lemma 8.4 when $X = S^{2n} = \mathbb{C}^n \cup \{\infty\}$, under complex conjugation and $n > 0$. Since $S^{2n}/\mathbb{Z}_2 = S^n \# \mathbb{P}^{n-1}_{\mathbb{R}}$, one has $\tilde{H}_k(S^{2n}/\mathbb{Z}_2;\mathbb{Z}) \cong \tilde{H}_k(S^n\#\mathbb{P}^{n-1}_{\mathbb{R}};\mathbb{Z}) \cong \tilde{H}_{k-n-1}(\mathbb{P}^{n-1}_{\mathbb{R}};\mathbb{Z})$. Hence the spaces $S^{2n}/\mathbb{Z}_2$ satisfy the hypothesis of Theorem A.5 and we obtain a canonical splitting

$$\begin{aligned}\mathcal{Z}_0(S^{2n})^{av}_o = \mathcal{Z}_0(S^{2n}/\mathbb{Z}_2)_o &\cong \prod_{k=0}^{2n} K(\tilde{H}_k(S^{2n}/\mathbb{Z}_2;\mathbb{Z}),\ k) \\ &\cong \prod_{k=n+1}^{2n} K(\tilde{H}_{k-n-1}(\mathbb{P}^{n-1}_{\mathbb{R}};\mathbb{Z}),\ k).\end{aligned}$$

Together with (8.6) and Proposition 8.1 this splitting yields the following.

Theorem 8.6. *The group of averaged cycles of codimension q and degree 0 in $\mathbb{P}^N_{\mathbb{C}}$ is connected and has a canonical splitting into products of Eilenberg-MacLane spaces*

$$\mathcal{Z}^q(\mathbb{P}^N_{\mathbb{C}})^{av}{}_o \cong \prod_{n=0}^{q}\prod_{k=1}^{n} K\left(H_{k-1}(\mathbb{P}^{n-1}_{\mathbb{R}};\mathbb{Z}),\ k+n\right).$$

The homotopy groups $\pi_*(\mathcal{Z}^q(\mathbb{P}^N_{\mathbb{C}})^{av})$ have the structure of a bigraded abelian group $\pi_*(\mathcal{Z}^q(\mathbb{P}^N_{\mathbb{C}})^{av}) \cong \bigoplus_{n,k\geq 0} I^{av}_{n,k}$ where

$$I^{av}_{n,k} \equiv \pi_{n+k}\left\{\mathcal{Z}_0(\mathbb{P}^n_{\mathbb{C}}/\mathbb{P}^{n-1}_{\mathbb{C}})_o\right\}^{av} \cong \tilde{H}_{k-1}(\mathbb{P}^{n-1}_{\mathbb{R}};\mathbb{Z}),$$

for $n+k>0$, and $I_{0,0} = 2\mathbb{Z}$. In other words,

$$I^{av}_{n,k} = \begin{cases} 2\mathbb{Z} & , \text{ if } n=k=0; \\ 0 & , \text{ if } k \text{ is odd, or } k>n, \text{ or } n>q; \\ \mathbb{Z} & , \text{ if } k=n\leq q \text{ and } k\geq 2 \text{ is even;} \\ \mathbb{Z}_2 & , \text{ if } k<n\leq q \text{ and } k\geq 2 \text{ is even.} \end{cases}$$

This is just Theorem 3.4. Combining it with (8.8) and (8.9) proves Theorem 3.3.

Remark 8.7. note that the results in this section prove in particular that:

(1) The inclusion $i : \mathcal{Z}^q(\mathbb{P}^N_{\mathbb{C}})^{av} \hookrightarrow \mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}^N_{\mathbb{C}})$ induces an inclusion of homotopy groups as direct summands;
(2) The inclusion $\mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}^N_{\mathbb{C}}) \hookrightarrow \mathcal{Z}^{q+1}_{\mathbb{R}}(\mathbb{P}^{N+1}_{\mathbb{C}})$ induced by the inclusion of $\mathbb{P}^N_{\mathbb{C}}$ as a linear subspace of $\mathbb{P}^{N+1}_{\mathbb{C}}$ induces an inclusion of homotopy groups as direct summands.

We now present an alternative approach to prove Theorem 3.3. Let $X = |X_\bullet|$ be the geometric realization of a simplicial set $X_\bullet$ on which the group $\mathbb{Z}_2$ acts simplicially, and let $\mathbb{Z}(X_\bullet)$ denote the simplicial free abelian group generated by $X_\bullet$. Note that $\mathbb{Z}(X_\bullet)$ becomes naturally a simplicial $\mathbb{Z}_2$-module under the induced action $\sigma : \mathbb{Z}(X_\bullet) \to \mathbb{Z}(X_\bullet)$.

If $\mathbb{Z}_2$-Mod and Ab denote the categories of $\mathbb{Z}_2$-modules and abelian groups, respectively, then the functors

$$Ker(1-\sigma)\ :\ \mathbb{Z}_2\text{-}Mod \to Ab \quad \text{and} \quad Im(1+\sigma)\ :\ \mathbb{Z}_2\text{-}Mod \to Ab$$

are additive functors which send a $\mathbb{Z}_2$-module M to M^{fix} and M^{av}, respectively.

Lemma 8.8. *With $X_\bullet$ as above:*

(1) *One has a natural equivariant identification $\mathcal{Z}_0(X) = |\mathbb{Z}(X_\bullet)|$.*
(2) *Under this identification one has $\mathcal{Z}_{0,\mathbb{R}}(X) = |\mathbb{Z}(X_\bullet)^{fix}|$ and $\mathcal{Z}_0(X)^{av} = |\mathbb{Z}(X_\bullet)^{av}|$. In other words, the real and averaged 0-cycles on X are obtained as the geometric realization of the simplicial abelian groups obtained by applying the functors $Ker(1-\sigma)$ and $Im(1+\sigma)$ to $\mathbb{Z}(X_\bullet)$.*

Proof. We leave the proof of this rather straightforward fact to the reader. □

Proof. [of Theorem 3.3 (2nd version)] Let $S^{1,1}$ denote the simplicial set with two zero-simplices 0 and ∞ (∞ is the base point) and two non-degenerate 1-simplices. Give $S^{1,1}$ the trivial $\mathbb{Z}_2$ action and note that its geometric realization is the circle seen as the one-point compactification of the trivial representation $\mathbb{R}$. Now, let $S^{0,1}$ denote the same simplicial set with the $\mathbb{Z}_2$ action that interchanges the two 1-simplices and keeps the 0-simplices fixed. The geometric realization of the latter is the one-point compactification of the sign representation. Denote $S^{n,n} \stackrel{\text{def}}{=} (S^{1,1})^{\wedge n}$ and $S^{0,n} \stackrel{\text{def}}{=} (S^{0,1})^{\wedge n}$.

It follows from Proposition 8.2 that the theorem is proven once we compute the homotopy groups of $\mathcal{Z}_{0,\mathbb{R}}(S^{2n})_o$, where S^{2n} is the one-point compactification of $\mathbb{C}^n$. It follows from Lemma 8.8 that $\mathcal{Z}_{0,\mathbb{R}}(S^{2n})_o = |\widetilde{\mathbb{Z}}(S^{n,n} \wedge S^{0,n})|$, where, for any equivariant simplicial set $X_\bullet$, $\widetilde{\mathbb{Z}}(X_\bullet)$ denotes the kernel of the natural augmentation $\mathbb{Z}(X_\bullet) \to \mathbb{Z}$.

Remark 8.9. Given $X_\bullet$ as above, the homotopy groups of $|\widetilde{\mathbb{Z}}(X_\bullet)|$ are obtained as the homology of the normalized chain complex $\widetilde{\mathbb{Z}}_{norm}(X_\bullet)$. Similarly, the homotopy groups of $|\widetilde{\mathbb{Z}}(X_\bullet)^{fix}|$ are given by the homology of the complex $\widetilde{\mathbb{Z}}_{norm}(X_\bullet)^{fix}$, which is easily seen to be the normalized chain complex associated to the simplicial abelian group $\widetilde{\mathbb{Z}}(X_\bullet)^{fix}$.

Notice that the normalized chain complexes of $\mathbb{Z}[\mathbb{Z}_2]$-modules $\widetilde{\mathbb{Z}}_{norm}(S^{n,n})$ and $\widetilde{\mathbb{Z}}_{norm}(S^{0,n})$ both have the form $C_n \to C_{n-1} \to \cdots \to C_0$, and both have just one homology group in degree n. In the first case, this group is simply $\mathbb{Z}$ with the trivial $\mathbb{Z}_2$-action, and in the latter the n-th homology is $\mathbb{Z}(n)$, i.e. the group $\mathbb{Z}$ with $\mathbb{Z}_2$ action given by multiplication by $(-1)^n$.

It follows that one has an equivariant homotopy equivalence

$$\widetilde{\mathbb{Z}}_{norm}(S^{n,n}) \simeq \mathbb{Z}[-n], \tag{8.12}$$

where $\mathbb{Z}$ is seen as a $\mathbb{Z}[\mathbb{Z}_2]$-complex concentrated in degree zero, and $\mathbb{Z}[-n]$ is its usual shift.

As to $\widetilde{\mathbb{Z}}_{norm}(S^{0,n})$, first observe that all terms C_i are free $\mathbb{Z}[\mathbb{Z}_2]$-modules, except for C_0, since the only non-degenerate simplices in $S^{0,n}$ fixed by the action of $\mathbb{Z}_2$ are zero dimensional. Now, denote $A^i = Hom_{\mathbb{Z}}(C_{n-i}, \mathbb{Z})$ and observe that one obtains an exact sequence

$$A^n \to \cdots \to A^0 \to \mathbb{Z}(n) \to 0,$$

where the A^i's are free, except in dimension n. Here, we use the fact that one has isomorphisms of $\mathbb{Z}[\mathbb{Z}_2]$-modules $Hom_{\mathbb{Z}}(\mathbb{Z}[\mathbb{Z}_2], \mathbb{Z}) \cong \mathbb{Z}[\mathbb{Z}_2]$ and $Hom_{\mathbb{Z}}(\mathbb{Z}(n), \mathbb{Z}) \cong \mathbb{Z}(n)$. It follows that this exact sequence is a truncated (at level n) projective resolution of $\mathbb{Z}(n)$ in the category of $\mathbb{Z}[\mathbb{Z}_2]$-modules. Applying the functor $Hom_{\mathbb{Z}_2}(\mathbb{Z}_2,-)$, and using the fact that $Hom_{\mathbb{Z}_2}(\mathbb{Z}_2,M)=M^{fix}$, for any $\mathbb{Z}_2$-module, it follows that the homology of $\widetilde{\mathbb{Z}}_{norm}(S^{0,n})^{fix}$ is given by:

$$
(8.13) \qquad H_k(\widetilde{\mathbb{Z}}_{norm}(S^{0,n})^{fix}) \;=\; \begin{cases} H^{n-k}(\mathbb{Z}_2,\mathbb{Z}(i)) & ,\ \text{for } 0\le k\le n \\ 0 & ,\ \text{for } k>n \text{ or } k<0. \end{cases}
$$

The (equivariant) Künneth formula gives an equivariant homotopy equivalence $\widetilde{\mathbb{Z}}_{norm}(S^{n,n}\wedge S^{0,n}) \;\cong\; \widetilde{\mathbb{Z}}_{norm}(S^{n,n})\otimes\widetilde{\mathbb{Z}}_{norm}(S^{0,n})$ which, together with (8.12) gives an equivariant homotopy equivalence

$$
\widetilde{\mathbb{Z}}_{norm}(S^{n,n}\wedge S^{0,n}) \;\cong\; \widetilde{\mathbb{Z}}_{norm}(S^{0,n})[-n],
$$

hence $\widetilde{\mathbb{Z}}_{norm}(S^{n,n}\wedge S^{0,n})^{fix} \;\cong\; \widetilde{\mathbb{Z}}_{norm}(S^{0,n})^{fix}[-n]$. This equivalence, together with (8.13), gives the desired computation:

$$
\pi_{n+k}(\mathcal{Z}_{0,\mathbb{R}}(S^{2n})) = H_{n+k}(\widetilde{\mathbb{Z}}_{norm}(S^{0,n})^{fix}[-n]) \;=\; H_k(\widetilde{\mathbb{Z}}_{norm}(S^{0,n})^{fix}).
$$

These cohomology groups are well-known and seen to coincide with the groups $I_{n,k}$. This concludes the alternative proof of Theorem 3.3. □

We now sketch an answer to Question 5.3, using the techniques developed in the proof above. The inclusion $\mathcal{Z}_{0,\mathbb{R}}(S^{2n})_o \to \mathcal{Z}_0(S^{2n})_o$ is a homomorphism of topological abelian groups induced by a homomorphism of corresponding simplicial abelian groups

$$
(8.14) \qquad \widetilde{\mathbb{Z}}(S^{n,n}\wedge S^{0,n})^{fix} \to \widetilde{\mathbb{Z}}(S^{2n,2n}).
$$

In order to understand this map, we invoke the Dold-Kan correspondence which states that the normalized chain complex functor gives equivalence between the categories of simplicial abelian groups and chain complexes in non-negative degrees. We have seen that one has a quasiisomorphism $\widetilde{\mathbb{Z}}_{norm}(S^{2n,2n})\simeq\mathbb{Z}[-2n]$, and (8.13) gives quasiisomorphisms

$$
\widetilde{\mathbb{Z}}_{norm}(S^{n,n}\wedge S^{0,n})^{fix} \simeq \begin{cases} \mathbb{Z}[-2n] \;\oplus_{i=0}^{n/2-1}\; \mathbb{Z}_2[-n-2i] & ,\text{if } n \text{ is even} \\ \oplus_{i=0}^{(n-1)/2}\; \mathbb{Z}_2[-n-2i] & ,\text{if } n \text{ is odd.} \end{cases}
$$

Under the Dold-Kan correspondence, the map (8.14) is given by a homomorphism of normalized complexes, and these homomorphisms are classified by Ext groups (over $\mathbb{Z}$).

In the even case ($n>0$) the map from $\mathbb{Z}[-2n]$ to itself is the identity, since the generator of $\pi_{2n}(\mathcal{Z}_0(S^{2n})_o)$ can be obtained using only real cycles. The other components, given by maps $\mathbb{Z}[-n-2i]\to\mathbb{Z}[-2n]$, are trivial since they are elements in $Ext^{n-2i}_{\mathbb{Z}}(\mathbb{Z}_2,\mathbb{Z})=0$.

In the odd case, the only component which may not be zero comes from the map $\mathbb{Z}[2n-1]\to\mathbb{Z}[2n]$. Following carefully the computation of the equivariant homotopy type of $\widetilde{\mathbb{Z}}_{norm}(S^{0,n})$ one obtains that this cohomology class is precisely $\beta(\widetilde{\imath}_{2n-1})$.

§9. The ring structure. The algebraic join of cycles, defined for example in [L$_1$], [L$_2$], and [FM] is equivariant with respect to conjugation and defines biadditive pairings $\mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}^n)\wedge\mathcal{Z}^{q'}_{\mathbb{R}}(\mathbb{P}^{n'})\longrightarrow\mathcal{Z}^{q+q'}_{\mathbb{R}}(\mathbb{P}^{n+n'+1})$. This induces a pairing

$$
\#:\mathcal{Z}^{\infty}_{\mathbb{R}}\wedge\mathcal{Z}^{\infty}_{\mathbb{R}}\longrightarrow\mathcal{Z}^{\infty}_{\mathbb{R}},
$$

where $\mathcal{Z}_{\mathbb{R}}^{\infty} = \lim_{n,q\to\infty} \mathcal{Z}_{\mathbb{R}}^q(\mathbb{P}^n)$, which makes $\mathcal{Z}_{\mathbb{R}}^{\infty}$ an E_∞-ring space. (See §6.) The induced map

$$\pi_*\mathcal{Z}_{\mathbb{R}}^{\infty} \otimes \pi_*\mathcal{Z}_{\mathbb{R}}^{\infty} \longrightarrow \pi_*\mathcal{Z}_{\mathbb{R}}^{\infty} \tag{9.1}$$

makes $\pi_*\mathcal{Z}_{\mathbb{R}}^{\infty}$ a graded ring. In this section we shall compute this ring and give explicit representatives for the generators.

To set the background we recall two analogous cases. Let

$$\mathcal{Z}^{\infty} = \lim_{n,q\to\infty} \mathcal{Z}^q(\mathbb{P}^n) \qquad \text{and} \qquad \widetilde{\mathcal{Z}}_{\mathbb{R}}^{\infty} = \lim_{n,q\to\infty} \widetilde{\mathcal{Z}}_{\mathbb{R}}^q(\mathbb{P}^n).$$

These are E_∞-ring spaces as seen in [BLLMM], and their homotopy groups form graded rings. Results from [FM] establish an isomorphism

$$\pi_*\mathcal{Z}^{\infty} \cong \mathbb{Z}[s] \tag{9.2}$$

where s corresponds to the generator of $\pi_2\mathcal{Z}^{\infty} \cong \mathbb{Z}$. Results of [Lam] show that

$$\pi_*\widetilde{\mathcal{Z}}_{\mathbb{R}}^{\infty} \cong \mathbb{Z}_2[y] \tag{9.3}$$

where y corresponds to the generator of $\pi_1\widetilde{\mathcal{Z}}_{\mathbb{R}}^{\infty} \cong \mathbb{Z}_2$. The main result of this section is the following theorem which neatly organizes the additive results of §8. Let $\mathcal{Z}_{\text{av}}^{\infty} \subset \mathcal{Z}_{\mathbb{R}}^{\infty}$ be the subspace defined by taking the limits of the subgroups $\mathcal{Z}_{\text{av}}^q(\mathbb{P}^n) \subset \mathcal{Z}_{\mathbb{R}}^q(\mathbb{P}^n)$ as above. Note that the join of an averaged cycle with a fixed cycle is again an averaged cycle.

Theorem 9.1. *There is a ring isomorphism*

$$\pi_*\mathcal{Z}_{\mathbb{R}}^{\infty} \cong \mathbb{Z}[x,y]/(2y) \tag{9.4}$$

where x corresponds to the generator of $\pi_4\mathcal{Z}_{\mathbb{R}}^{\infty} \cong \mathbb{Z}$ and y corresponds to the generator of $\pi_1\mathcal{Z}_{\mathbb{R}}^{\infty} \cong \mathbb{Z}_2$, and where $(2y)$ denotes the principal ideal in the polynomial ring generated by $2y$. Furthermore, under this isomorphism the ideal $\pi_*\mathcal{Z}_{\text{av}}^{\infty} \subset \pi_*\mathcal{Z}_{\mathbb{R}}^{\infty}$ corresponds to the ideal

$$\pi_*\mathcal{Z}_{\text{av}}^{\infty} \cong (2,x) \tag{9.5}$$

generated by 2 and x.

Proof. To begin we introduce a doubly indexed filtration on $\pi_*\mathcal{Z}_{\mathbb{R}}^{\infty}$ and show that it is compatible with the multiplication (9.1). Consider the direct sum $\mathbb{C}^\infty$ with coordinates $(z_0, z_1, z_2, \dots)$; $z_j = x_j + iy_j$, and for each n set $\mathbb{C}^{n+1} = \{z \in \mathbb{C}^\infty : z_j = 0 \text{ for } j > n\}$. For each k, $0 \le k \le n$, we consider the conjugation invariant subspace

$$V^{n,k} = \{z \in \mathbb{C}^{n+1} \ : \ y_j = 0 \text{ for } j \ge k\} \cong \mathbb{C}^k \oplus \mathbb{R}^{n+1-k}.$$

and set

$$\mathbb{P}^{n,k} = \pi(V^{n,k} - \{0\})$$

where $\pi : \mathbb{C}^{n+1} - \{0\} \longrightarrow \mathbb{P}^n$ is the projection. Note that $V^{n,k} \subset V^{n',k'}$ if $n \leq n'$ and $k \leq k'$, and that

$$\dim_{\mathbb{R}} \mathbb{P}^{n,k} \ = \ n + k.$$

Definition. Let $\mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}^{n,k}) \subset \mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}^n)$ denote the subgroup generated by effective cycles c for which

$$\dim_{\mathbb{R}} \left(|\widetilde{c}| \cap V^{n,k}\right) \ \geq \ \dim_{\mathbb{C}} \left(|\widetilde{c}|\right) \tag{9.6}$$

where $|\widetilde{c}| = \pi^{-1}(|c|) \subset \mathbb{C}^{n+1}$ denotes the homogeneous cone corresponding to the support $|c|$ of c. The inclusions $\mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}^{n,k}) \subset \mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}^n) \subset \mathcal{Z}^\infty_{\mathbb{R}}$ induce homomorphisms

$$\pi_* \mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}^{n,k}) \ \longrightarrow \ \pi_* \mathcal{Z}^\infty_{\mathbb{R}} \tag{9.7}$$

Observation 9.2 The image of the homomorphism (9.7) remains constant under continuous deformations of $\mathbb{C}^n$ through Real subspaces of $\mathbb{C}^\infty$

Observation 9.3 The homomorphism

$$\pi_* \mathcal{Z}^n_{\mathbb{R}}(\mathbb{P}^{n,k}) \longrightarrow \pi_* \mathcal{Z}^\infty_{\mathbb{R}}$$

is injective. This follows from the results in 8.6 concerning 0- cycles.

Observation 9.4 Taking algebraic suspension by adding coordinates on the left gives a commutative diagram

$$\begin{array}{ccc} \pi_* \mathcal{Z}^n_{\mathbb{R}}(\mathbb{P}^{n,k}) & \xrightarrow{\ \ j\ \ } & \pi_* \mathcal{Z}^\infty_{\mathbb{R}} \\ \widetilde{\Sigma}^\ell_* \big\downarrow & & \| \big\downarrow \Sigma^\ell_* \\ \pi_* \mathcal{Z}^n_{\mathbb{R}}(\mathbb{P}^{n+\ell,k+\ell}) & \xrightarrow{\ \ j_\ell\ \ } & \pi_* \mathcal{Z}^\infty_{\mathbb{R}} \end{array} \tag{9.8}$$

where j is injective by 9.3, Σ^ℓ_* is an isomorphism, and $\widetilde{\Sigma}^\ell_*$, induced by the restriction of the suspension map, is therefore also injective.

We set

$$\mathcal{F}^{n,k} \ \stackrel{\text{def}}{=} \ j \pi_* \mathcal{Z}^n_{\mathbb{R}}(\mathbb{P}^{n,k})$$

and note that $\mathcal{F}^{n,k}$ gives a bifiltration of $\pi_* \mathcal{Z}^\infty_{\mathbb{R}}$, namely

$$\mathcal{F}^{n,k} \subset \mathcal{F}^{n',k'} \qquad \text{if } n \leq n' \ \text{ and } \ k \leq k'.$$

Proposition 9.5 *The homomorphism $\widetilde{\Sigma}^\ell_*$ in (9.8) is an isomorphism. Consequently,*

$$\mathcal{F}^{n,k} \ = \ Im(j_\ell) \qquad \textit{for all } \ell > 0.$$

Proof. In fact the map $\widetilde{\Sigma}_*^{\ell} : \mathcal{Z}^n_{\mathbb{R}}(\mathbb{P}^{n,k}) \to \mathcal{Z}^n_{\mathbb{R}}(\mathbb{P}^{n+\ell,k+\ell})$ is a homotopy equivalence. To see this one repeats the arguments of [Lam] which prove that $\Sigma_* : \mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}^n) \longrightarrow \mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}^{n+1})$ is a homotopy equivalence, and one notes that all steps preserve the subgroups $\mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}^{n+1,*})$. More specifically, there are two fundamental constructions in this proof: pulling to the normal cone and "magic fans".

We begin with pulling to the normal cone. Let's introduce homogeneous coordinates $(z,x) \in \mathbb{C}^k \oplus \mathbb{R}^{n+1-k} = V^{n,k} \subset \mathbb{C}^{n+1}$ and $(\xi, z, x) \in \mathbb{C} \oplus \mathbb{C}^k \oplus \mathbb{R}^{n+1-k} = V^{n+1,k+1} \subset \mathbb{C}^{n+2}$. Consider the multiplicative flow φ_t on $\mathbb{P}^{n+1}$ defined in homogeneous coordinates by $\varphi_t(\xi,z,x) = (t\xi, z, x)$ for $t \in \mathbb{R}^+$. This flow induces "pulling to the normal cone" in [Lam]. It evidently preserves condition (96) above, and therefore preserves the subgroup of algebraic cycles $\mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}^{n+1,k+1})$.

In the "magic fan" construction one adds a new coordinate giving $(\eta, \xi, z, x) \in \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}^k \oplus \mathbb{R}^{n+1-k} = \mathbb{C} \oplus V^{n+1,k+1}$. To each homogeneous polynomial $f(\eta,\xi,z,x)$ with real coefficients one constructs a transformation $\Phi_f : \mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}^{n+1}) \to \mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}^{n+1})$ by setting $\Phi_f(c) = (\pi_1)_*(\pi_0^* c \bullet D_f)$ where D_f is the divisor of f, and π_0, π_1 are projections $\mathbb{P}^{n+2}_{\mathbb{C}} \cdots > \mathbb{P}^{n+1}_{\mathbb{C}}$ with vertices $(1,0,0,\ldots,0)$ and $(1,1,0,\ldots,0)$ respectively. We need to check that this construction preserves the subgroups $\mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}^{n+1,k+1})$. For this let $\widetilde{Y} \subset \mathbb{C}^{n+2}$ denote the homogeneous cone of a projective variety $Y \subset \mathbb{P}^{n+1}$. It will suffice to show that

$$(9.9) \qquad \dim_{\mathbb{R}}\left(\widetilde{Y} \cap V^{n+1,k+1}\right) \;\le\; \dim_{\mathbb{R}}\left(\widetilde{(\Phi_f Y)} \cap V^{n+1,k+1}\right)$$

To see this consider a point $a \in Y$ with homogeneous coordinates $(\xi,z,x) \in \widetilde{Y} \cap V^{n+1,k+1}$. Let $\eta_1, \ldots, \eta_d$ be the zeros of the polynomial $q(t) = f(t,\xi,z,x)$. Then $\pi_0^{-1}(a) \cap D_f$ consists of the d points with homogeneous coordinates (η_j, ξ, z, x), $j = 1, \ldots, d$, and $\pi_1\left(\pi_0^{-1}(a) \cap D_f\right)$ is the union of the d points with homogeneous coordinates $(\eta_j - \xi, z, x)$, $j = 1, \ldots, d$. Note that each of these points again lies in $V^{n+1,k+1}$. It follows that condition (9.9) holds as claimed. Therefore the arguments of [Lam] apply without change to show that $\widetilde{\Sigma}$ is a homotopy equivalence, and we are done. $\square$

The analogues of 9.2 – 9.5 apply also to the averaged cycles:

$$\mathcal{Z}^q_{\text{av}}(\mathbb{P}^{n,k}) = \mathcal{Z}^q_{\mathbb{R}}(\mathbb{P}^{n,k}) \cap \mathcal{Z}^q_{\text{av}}(\mathbb{P}^n)$$

One obtains a bifiltration $\mathcal{F}^{n,k}_{\text{av}}$ of $\pi_* \mathcal{Z}^\infty_{\text{av}}$ where the n-filtration agrees with that of Theorem 8.5. Under the isomorphism $\pi_* \mathcal{Z}^\infty_{\text{av}} \cong \widetilde{H}_*(X^\infty; \mathbb{Z})$ deduced in §8 (See 8.3), consider the classes

$$(9.10) \qquad \theta_{n,k} \;=\; [\mathbb{P}^{n,k}/\mathbb{Z}_2] \in \widetilde{H}_{n+k}(X^\infty;\mathbb{Z}) \cong \pi_{n+k}\mathcal{Z}^\infty_{\text{av}} \qquad \text{for} \quad 0 < k \le n.$$

Note that the intersection of $\mathbb{P}^{n,k}$ with the affine coordinate chart $\mathbb{P}^n_{\mathbb{C}} - \mathbb{P}^{n-1}_{\mathbb{C}} = \mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n$ is exactly $\mathbb{R}^n \oplus i\mathbb{R}^k$. Therefore, the image of $\theta_{n,k}$ in the homology of $X^n/X^{n-1} = S^{2n}/\mathbb{Z}_2$ is the class $S^n \# \mathbb{P}^{k-1}_{\mathbb{R}}$. Hence this image generates the group $I^{av}_{n,k}$ in the bigrading established in Theorem 8.5.

It follows that

$$\mathcal{F}^{n,k}_{\text{av}} \;=\; \text{span}_{\mathbb{Z}}\{\theta_{n',k'} \,:\, n' \leq n \;\text{ and }\; k' \leq k\},$$

and furthermore that under the injection $\pi_*\mathcal{Z}^\infty_{\text{av}} \hookrightarrow \pi_*\mathcal{Z}^\infty_{\mathbb{R}}$ we have that

$$\mathcal{F}^{n,k}_{\text{av}} \;=\; \mathcal{F}^{n,k} \cap \pi_*\mathcal{Z}^\infty_{\text{av}}. \tag{9.11}$$

In particular, we deduce that for $0 < k \leq n$

$$\begin{aligned} I^{av}_{n,k} &\stackrel{\text{def}}{=} \mathcal{F}^{n,k}_{\text{av}} \cap \pi_{n+k}\mathcal{Z}^\infty_{\text{av}} = \langle \theta_{n,k} \rangle \\ &\cong \mathcal{F}^{n,k}_{\text{av}}/\mathcal{F}^{n-1,k}_{\text{av}} \oplus \mathcal{F}^{n,k-1}_{\text{av}} \end{aligned}$$

and $I^{av}_{n,0} = \{0\}$. For $0 \leq k \leq n$ set $I_{n,k} \equiv \mathcal{F}^{n,k} \cap \pi_{n+k}\mathcal{Z}^\infty_{\mathbb{R}}$. From Theorem 8.2 we know that $I_{n,0} \cong \pi_n\{\mathcal{Z}_0(\mathbb{P}-\mathbb{C}^n/\mathbb{P}-\mathbb{C}^{n-1})_o\}^{fix} = \mathbb{Z}_2$. Let $\theta_{n,0}$ denote the generator of $I_{n,0}$. (an explicit representative will be given later). Then we have that for $0 \leq k \leq n$

$$\begin{aligned} I_{n,k} &\stackrel{\text{def}}{=} \mathcal{F}^{n,k} \cap \pi_{n+k}\mathcal{Z}^\infty_{\mathbb{R}} = \langle \theta_{n,k} \rangle \\ &\cong \mathcal{F}^{n,k}/\mathcal{F}^{n-1,k} \oplus \mathcal{F}^{n,k-1} \end{aligned}$$

It is useful to picture the graded peices $I_{n,k}$ on the (n,k)-coordinate grid.

$k\uparrow$											...	
										$\mathbb{Z}$	...	
									0	0	...	
								$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	...	
							0	0	0	0	...	
						$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	...	
					0	0	0	0	0	0	...	
				$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	...	
			0	0	0	0	0	0	0	0	...	
		$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	...	
	0	0	0	0	0	0	0	0	0	0	...	
$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	...	$\underset{n}{\longrightarrow}$

The graded subgroup $I^{av}_{n,k} \subset I_{n,k}$ looks like this:

$k\uparrow$											...	
										$\mathbb{Z}$	...	
									0	0	...	
								$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	...	
							0	0	0	0	...	
						$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	...	
					0	0	0	0	0	0	...	
				$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	...	
			0	0	0	0	0	0	0	0	...	
		$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	...	
	0	0	0	0	0	0	0	0	0	0	...	
$2\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	...	$\underset{n}{\longrightarrow}$

Note that $I_{n,k} = I^{av}_{n,k}$ for $k > 0$ and $\widetilde{I}_{*,*} \equiv I_{*,*}/I^{av}_{*,*}$ is simply:

$$
\begin{array}{cccccccccccccc}
k\uparrow & & & & & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \\
\mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \dots & \underset{n}{\longrightarrow}
\end{array}
$$

Our main observation here is the following.

Proposition 9.6. *The filtrations $\mathcal{F}^{n,k}$ and $\mathcal{F}^{n,k}_{\text{av}}$ are compatible with the join pairing, i.e.,*

$$\#\left(\mathcal{F}^{n,k} \otimes \mathcal{F}^{n',k'}\right) \subset \mathcal{F}^{n+n',k+k'} \qquad \text{and} \qquad \#\left(\mathcal{F}^{n,k}_{\text{av}} \otimes \mathcal{F}^{n',k'}_{\text{av}}\right) \subset \mathcal{F}^{n+n',k+k'}_{\text{av}}$$

Proof. By Observation 9.2 the join gives a well defined homomorphism

$$\pi_* \mathcal{Z}^n_{\mathbb{R}}(\mathbb{P}^{n,k}) \otimes \pi_* \mathcal{Z}^{n'}_{\mathbb{R}}(\mathbb{P}^{n',k'}) \longrightarrow \pi_* \mathcal{Z}^{n+n'}_{\mathbb{R}}(\mathbb{P}^{n+n'+1,k+k'+1}).$$

Pushing into $\pi_* \mathcal{Z}^\infty_{\mathbb{R}}$ and applying 9.5 gives the result for $\mathcal{F}^{*,*}$. The argument for $\mathcal{F}^{*,*}_{\text{av}}$ is similar. $\square$

Lemma 9.7. *The generators $\theta_{2,2} \in \pi_4 \mathcal{Z}^\infty_{\mathbb{R}} \cong \pi_4 \mathcal{Z}^\infty_{\text{av}} \cong \mathbb{Z}$ and $\theta_{1,0} \in \pi_1 \mathcal{Z}^\infty_{\mathbb{R}}$ have the property that*

$$\theta_{2,2} \in \mathcal{F}^{2,2} \qquad \text{and} \qquad \theta_{1,0} \in \mathcal{F}^{1,0}$$

Proof. Recall $X^q \equiv \mathbb{P}^q_{\mathbb{C}}/\mathbb{Z}_2$. (See Cor. 8.4.) Under the isomorphisms $\pi_4 \mathcal{Z}^q_{\text{av}} \cong \pi_4 \mathcal{Z}_0(X^q) \cong H_4(X^q;\mathbb{Z}) \cong \mathbb{Z}$, the generator $\theta_{2,2}$ corresponds to the class $[X^2] \in H_4(X^q;\mathbb{Z})$ for any $q \geq 2$. Thus $\theta_{2,2} \in \mathcal{F}^{2,2}$. For the second assertion recall that the generator of $\pi_1 \mathcal{Z}^\infty_{\mathbb{R}}$ is given by the map $S^1 \longrightarrow \mathcal{Z}_0(\mathbb{P}^1_{\mathbb{R}}) \subset \mathcal{Z}_0(\mathbb{P}^1_{\mathbb{C}})$ sending $t \mapsto t - t_0$ for $t \in \mathbb{P}^1_{\mathbb{R}} \cong S^1$ (where $t_0 \in \mathbb{P}^1_{\mathbb{R}}$ is a base point). $\square$

Proposition 9.8.

A) $\theta^k_{2,2} = \theta_{2k,2k}$ *in* $\pi_{4k} \mathcal{Z}^\infty_{\text{av}}$.
B) $\theta^k_{2,2} \cdot \theta^\ell_{1,0} \neq 0$ *in* $\pi_{4k+\ell} \mathcal{Z}^\infty_{\text{av}}$.

Proposition 9.8 $\Rightarrow$ Theorem 9.1. From results in §8 we have an exact sequence

$$0 \longrightarrow \pi_* \mathcal{Z}^\infty_{\text{av}} \longrightarrow \pi_* \mathcal{Z}^\infty_{\mathbb{R}} \xrightarrow{p_*} \pi_* \widetilde{\mathcal{Z}}^\infty_{\mathbb{R}} \longrightarrow 0 \tag{9.12}$$

and we know that the elements $p_*(\theta^\ell_{1,0})$, $\ell \geq 1$, give an additive basis of $\pi_* \widetilde{\mathcal{Z}}^\infty_{\mathbb{R}}$. In fact, p_* is a ring homomorphism and $p_*(\theta_{1,0})$ corresponds to y in (9.3).

We have seen that under the identification

$$\pi_* \mathcal{Z}^\infty_{\text{av}} \cong H_*(X^\infty;\mathbb{Z})$$

the bifiltration of $\pi_* \mathcal{Z}^\infty_{\text{av}}$ corresponds to the bifiltration of $H_*(X^\infty; \mathbb{Z})$ induced by the family of subspaces $\mathbf{P}^{n,k}/\mathbb{Z}_2$ in X^∞. From the results of §8 we have that

$$\mathcal{F}^{*,\text{odd}}_{\text{av}} = 0$$

and

$$\mathcal{F}^{2k+\ell,2k}_{\text{av}} \cap \pi_{4k+\ell}\mathcal{Z}^\infty_{\text{av}} = \begin{cases} \mathbb{Z}\cdot\theta_{2k,2k} & \text{if } \ell = 0 \\ \mathbb{Z}_2\cdot\theta_{2k+\ell,2k} & \text{if } \ell > 0. \end{cases}$$

Since

$$\theta^k_{2,2}\cdot\theta^\ell_{1,0} \in \mathcal{F}^{2k+\ell,2k}_{\text{av}} \cap \pi_{4k+\ell}\mathcal{Z}^\infty_{\text{av}},$$

the conclusion of Proposition 9.8 implies that $\theta^k_{2,2}\cdot\theta^\ell_{1,0} = \theta_{2k+\ell,2k}$. □

Proof of Proposition 9.8 We begin by constructing explicit representatives of $\theta_{1,0}$ and $\theta_{2,2}$.

Consider $S^1 = \mathbb{P}^1_{\mathbb{R}} \subset \mathbb{P}^1_{\mathbb{C}}$, the fixed- point set, and choose a base point $t_0 \in S^1$. Define

(9.13) $$\alpha : S^1 \longrightarrow \mathcal{Z}^1_{\mathbb{R}}(\mathbb{P}^1)$$

by

$$\alpha(t) = t - t_0.$$

This map clearly has filtration level (1,0) since the image is supported in $\mathbb{P}^1_{\mathbb{R}}$. One sees directly that under the projection $p_* : \mathcal{Z}^1_{\mathbb{R}}(\mathbb{P}^1) \longrightarrow \widetilde{\mathcal{Z}}^1_{\mathbb{R}}(\mathbb{P}^1_{\mathbb{C}}) = \mathcal{Z}_0(\mathbb{P}^1_{\mathbb{R}}) \otimes \mathbb{Z}_2$ the map $p_* \circ \alpha$ represents the generator of $\pi_1(\mathcal{Z}_0(\mathbb{P}^1_{\mathbb{R}})) \cong H_1(\mathbb{P}^1_{\mathbb{R}}; \mathbb{Z}_2) \cong \mathbb{Z}_2$. Hence α represents the non-zero class $\theta_{1,0}$ in $\pi_1\mathcal{Z}^\infty_{\mathbb{R}} = \mathbb{Z}_2$.

Consider $\mathbb{P}^1_{\mathbb{R}} \subset \mathbb{P}^1_{\mathbb{C}}$ as the "equator" and let $D^2 \subset \mathbb{P}^1_{\mathbb{C}}$ be the "upper hemisphere", (so $\mathbb{P}^1_{\mathbb{C}} = D^2 \cup \overline{D^2}$ where $\overline{(\cdot)}$ is the conjugation map). For each $n \geq 1$ we define a map

(9.14) $$\beta_n : (D^2)^n \longrightarrow \mathcal{Z}^n(\mathbb{P}^{2n-1}_{\mathbb{C}})$$

by

$$\beta_n(t_1, \dots, t_n) = (t_1 - \bar{t}_1)\#\dots\#(t_n - \bar{t}_n).$$

Note that $\beta_n(t_1, \dots, t_n) = 0$ if $t_j \in \partial D^2 = \mathbb{P}^1_{\mathbb{R}}$ for **any** j. Thus β_n descends to a map

$$\overline{\beta}_n : S^2 \wedge \cdots \wedge S^2 = S^{2n} \longrightarrow \mathcal{Z}^n(\mathbb{P}^{2n-1}_{\mathbb{C}}).$$

Note that $\overline{\beta_1(t)} = -\beta_1(t)$, that is β_1 maps into *anti-averaged* cycles. The join of two anti-averaged cycles in an averaged cycle. Since $\beta_n(t_1, \dots, t_n) = \beta_1(t_1)\#\dots\#\beta_n(t_n)$ we see that

$$\overline{\beta}_n : S^{2n} \longrightarrow \mathcal{Z}^n_{\text{av}}(\mathbb{P}^{2n-1}_{\mathbb{C}})$$

whenever n is even.

From Observation 9.5 we see that the class $[\overline{\beta}_2]$ of $\overline{\beta}_2 : S^4 \to \mathcal{Z}^2_{\text{av}}(\mathbb{P}^3) = \mathcal{Z}^2_{\text{av}}(\mathbb{P}^{3,3})$ in $\pi_4\mathcal{Z}^\infty_{\text{av}}$ has filtration level (2,2). It follows that the class $[\overline{\beta}_{2n}] = [\overline{\beta}_2]^n \in \pi_{4n}\mathcal{Z}^\infty_{\text{av}}$ has filtration level $(2n, 2n)$.

Lemma 9.9. *The map* $\overline{\beta}_n : S^{2n} \longrightarrow \mathcal{Z}^n(\mathbb{P}^{2n-1}_{\mathbb{C}})$ *represents the generator of* $\pi_{2n}\mathcal{Z}^\infty \cong \mathbb{Z}$.

Proof. Recall that $\beta_1 : (D^2, \partial D^2) \longrightarrow (\mathcal{Z}_0(\mathbb{P}^1_{\mathbb{C}}, 0)$ is given by $\beta_1(t) = t - \bar{t}$. To compute the class of $\overline{\beta}_1$ in $\pi_2\mathcal{Z}_0(\mathbb{P}^1_{\mathbb{C}}) = H_2(\mathbb{P}^1_{\mathbb{C}}; \mathbb{Z})$ we take the graph $\Gamma(\overline{\beta}_1)$ in $S^2 \times \mathbb{P}^1_{\mathbb{C}}$ and push it forward to $\mathbb{P}^1_{\mathbb{C}}$ (cf. [FL]). Now $\Gamma(\overline{\beta}_1)$ is an oriented cycle which is the union of two oriented disks $\Gamma_0 \cup \Gamma_1$. The disk Γ_0 is the graph of the identity map $D^2 \to D^2_+$ on the upper hemisphere with the canonical orientation from D^2; the disk Γ_1 is the graph of the conjugation map $D^2 \to D^2_-$ from the upper to the lower hemisphere with the orientation opposite the one given by D^2 due to the minus sign in $\overline{\beta}_1$. Note that $\Gamma_0 \cup \Gamma_1 \subset S^2 \times \mathbb{P}^1_{\mathbb{C}}$ is an oriented 2-sphere homeomorphic to $\mathbb{P}^1_{\mathbb{C}}$ under projection to the second factor.

This shows that $\overline{\beta}_1$ represents the generator s of $\pi_2\mathcal{Z}^\infty$. It follows that $\overline{\beta}_n = \overline{\beta}_1 \# \ldots \# \overline{\beta}_1$ represents $s^n \in \pi_{2n}\mathcal{Z}^\infty$, which is the generator by 9.2. □

Lemma 9.10. *For any* $N \geq 2n$, *the homomorphism* $I_* : \pi_{4n}\mathcal{Z}^{2n}_{\mathrm{av}}(\mathbb{P}^N_{\mathbb{C}}) \longrightarrow \pi_{4n}\mathcal{Z}^\infty$, *induced by the inclusion* $i : \mathcal{Z}^{2n}_{\mathrm{av}}(\mathbb{P}^N_{\mathbb{C}}) \hookrightarrow \mathcal{Z}^\infty$, *is an isomorphism.*

Proof. By the Algebraic Suspension Theorem [L$_1$], [LLM$_2$] it suffices to consider the map of 0-cycles $\mathcal{Z}^{2n}_{\mathrm{av}}(\mathbb{P}^{2n}_{\mathbb{C}}) \longrightarrow \mathcal{Z}^{2n}(\mathbb{P}^{2n}_{\mathbb{C}})$. Note that the composition

$$\mathcal{Z}^{2n}_{\mathrm{av}}(\mathbb{P}^{2n}_{\mathbb{C}}) \xrightarrow{i} \mathcal{Z}^{2n}(\mathbb{P}^{2n}_{\mathbb{C}}) \xrightarrow{\mathrm{av}} \mathcal{Z}^{2n}_{\mathrm{av}}(\mathbb{P}^{2n}_{\mathbb{C}})$$

where $\mathrm{av}(x) = x + \bar{x}$, is multiplication by 2, and so therefore is the composition

$$\begin{array}{ccccc} \pi_{4n}\mathcal{Z}^{2n}_{\mathrm{av}}(\mathbb{P}^{2n}_{\mathbb{C}}) & \xrightarrow{i_*} & \pi_{4n}\mathcal{Z}^{2n}(\mathbb{P}^{2n}_{\mathbb{C}}) & \xrightarrow{\mathrm{av}_*} & \pi_{4n}\mathcal{Z}^{2n}_{\mathrm{av}}(\mathbb{P}^{2n}_{\mathbb{C}}) \\ \| & & \| & & \| \\ \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} \end{array}$$

On the other hand the homomorphism av_* can be identified with the homomorphism $\rho_* : H_{4n}(\mathbb{P}^{2n}_{\mathbb{C}}; \mathbb{Z}) \longrightarrow H_{4n}(\mathbb{P}^{2n}_{\mathbb{C}}/\mathbb{Z}_2; \mathbb{Z})$ where $\rho : \mathbb{P}^{2n}_{\mathbb{C}} \longrightarrow \mathbb{P}^{2n}_{\mathbb{C}}/\mathbb{Z}_2$ is the quotient map. This map clearly sends the fundamental class $[\mathbb{P}^{2n}_{\mathbb{C}}]$ to $2[\mathbb{P}^{2n}_{\mathbb{C}}/\mathbb{Z}_2]$. Hence $\mathrm{av}_* = 2$ and so i_* must be an isomorphism. □

Corollary 9.11. *The map* $\overline{\beta}_2$ *represents the generator* $\theta_{2,2}$ *of* $\pi_4\mathcal{Z}^2_{\mathrm{av}}(\mathbb{P}^3_{\mathbb{C}})$. *Furthermore, for all* $n \geq 1$ *one has that* $\theta^n_{2,2} = \theta_{n,n}$.

Proof. The first assertion follows immediately from 9.9 and 9.10. For the second we recall that $[\overline{\beta}_{2n}] = [\overline{\beta}_2]^n$ in $\pi_* \mathcal{Z}^\infty_{\text{av}}$, and use (9.2). $\square$

This establishes part A) of Proposition 9.8. For part B) we invoke the following. Consider a continuous map $f : S^m \to \mathcal{Z}^q_{\text{av}}(\mathbb{P}^N_{\mathbb{C}})$ and let $\rho : \mathbb{P}^N_{\mathbb{C}} \to X^N$ be the projection. Then $\rho_* f(x) = 2\bar{f}(x)$ where $\bar{f} : S^m \to \mathcal{Z}^q(X^N)$ is a continuous map into cycles on X^N. Assume that f is well enough behaved to have a graph Γ_f in $S^m \times \mathbb{P}^N_{\mathbb{C}}$ (cf. [FL]). Then $(1 \times \rho_*)[\Gamma_f] = 2[\Gamma_{\bar{f}}]$ where $\Gamma_{\bar{f}}$ is a cycle on $S^m \times X^N$ which we will call the graph of $\bar{f}$. Let $\text{pr} : S^m \times X^N \to X^N$ be projection.

Lemma 9.12. *If $[\text{pr}_* \Gamma_{\bar{f}}] \neq 0$ in $H_*(X^N; \mathbb{Z})$, then $[f] \neq 0$ in $\pi_m \mathcal{Z}^q_{\text{av}}(\mathbb{P}^N_{\mathbb{C}})$.*

Proof. Suppose $f : \partial D^{m+1} \to \mathcal{Z}^q_{\text{av}}(\mathbb{P}^N_{\mathbb{C}})$ extends to a continuous map $F : D^{m+1} \to \mathcal{Z}^q_{\text{av}}(\mathbb{P}^N_{\mathbb{C}})$ which we may assume to have a graph. Then the integral chain $\text{pr}_* \Gamma_{\bar{F}}$ has boundary $\text{pr}_* \Gamma_{\bar{f}}$ in X^N. $\square$

To detect homology classes in X^N we will use the following.

Lemma 9.13. *Let $Z \subset X^N$ be an integral cycle of codimension $\ell < N$ defined by the oriented regular set of a real analytic subvariety. Let $\mathbb{P}^N_{\mathbb{R}} \subset X^N = \mathbb{P}^N_{\mathbb{C}} / \mathbb{Z}_2$ denote the singular set of X^N. Suppose there exists a compact oriented submanifold $Y^\ell \hookrightarrow X^N - \mathbb{P}^N_{\mathbb{R}}$ of dimension ℓ which meets the regular set of Z transversely in one point. Then $[Z] \neq 0$ in $H_{2N-\ell}(X^N; \mathbb{Z})$.*

Proof. Let $M = X^N - (\mathbb{P}^N_{\mathbb{R}})_\epsilon$ where $(\mathbb{P}^N_{\mathbb{R}})_\epsilon$ is a tubular neighborhood of $\mathbb{P}^N_{\mathbb{R}}$ whose closure does not meet Y^ℓ. Note that M is a smooth compact oriented manifold with boundary. The restriction $Z_\epsilon \equiv Z \cap M$ defines an integral cycle of codimension ℓ on $(M, \partial M)$, and Y^ℓ defines a cycle of dimension ℓ on M. The intersection hypothesis implies that $[Z_\epsilon] \neq 0$ in $H_{2N-\ell}(M, \partial M) \cong H_{2N-\ell}(X^N, \mathbb{P}^N_{\mathbb{R}})$. In the long exact sequence for the pair $(X^N, \mathbb{P}^N_{\mathbb{R}})$ we have

$$H_j(\mathbb{P}^N_{\mathbb{R}}) \longrightarrow H_j(X^N) \xrightarrow{r} H_j(X^N, \mathbb{P}^N_{\mathbb{R}}),$$

and it is clear from the construction that $r([Z]) = [Z_\epsilon]$. $\square$

To complete the proof of Proposition 9.8 B) we consider the map

$$f \equiv \overline{\beta}_2{}^n \alpha^\ell : S^{4n+2\ell} \longrightarrow \mathcal{Z}^{2n+\ell}(\mathbb{P}^{4n+\ell-1}_{\mathbb{C}})$$

where we may assume that ℓ is odd. This map can be coordinatized as follows. Choose affine coordinates $x_1, \ldots, x_{2n}, y_1, \ldots, y_\ell$ on $\mathbb{P}^1_{\mathbb{C}} \times \cdots \times \mathbb{P}^1_{\mathbb{C}}$ ($(2n+\ell)$- times) and restrict them to

$$\text{Im}(x_i) \geq 0 \qquad \text{and} \qquad \text{Im}(y_j) = 0 \qquad \text{for all } \ i, j.$$

Let $\lambda_{x_i} = \text{span}(1, x_i)$ and $\lambda_{y_j} = \text{span}(1, y_j)$ in $\mathbb{C}^2$. Then

$$f(x,y) = (\lambda_{x_1} - \lambda_{\overline{x_1}}) \# \ldots \# (\lambda_{x_{2n}} - \lambda_{\overline{x_{2n}}}) \# (\lambda_{y_1} - \lambda_0) \# \ldots \# (\lambda_{y_\ell} - \lambda_0).$$

Note that $f = \text{av} \circ \bar{f}$ where

$$\bar{f}(x,y) = \lambda_{x_1} \# (\lambda_{x_2} - \lambda_{\overline{x_2}}) \# \ldots \# (\lambda_{x_{2n}} - \lambda_{\overline{x_{2n}}}) \# (\lambda_{y_1} - \lambda_0) \# \ldots \# (\lambda_{y_\ell} - \lambda_0).$$

Consider the affine chart $\mathbb{C}^{4n+2\ell-1}$ on $\mathbb{P}(\mathbb{C}^2 \oplus \cdots \oplus \mathbb{C}^2) = \mathbb{P}(\mathbb{C}^{4n+2\ell})$ given be setting the first coordinate equal to 1. Let $\mathbb{C}^N/\mathbb{Z}_2$ be its image in X^N where $N = 4n + 2\ell - 1$. Then $Z = \mathrm{pr}_*(\Gamma_{\bar{f}})$ is an analytic cycle whose image in $\mathbb{C}^N/\mathbb{Z}_2$ is described as follows. Note that $\bar{f}$ expands into $2^{2n-1+\ell}$ factors. However those factors containing the constant λ_0 project to 0 for dimension reasons. Thus it suffices to consider the remaining 2^{2n-1} factors. Upstairs in $\mathbb{C}^N$ they are subsets of the form

$$Z_{\pm\pm\cdots\pm} \;=\; \bigcup\Big\{x_1 + \mathrm{span}\Big((0,1,x_2,0,...,0),(0,0,0,1,x_3,0,...,0),\ldots,(0,\ldots,0,1,x_{2n},0,...,0), (0,\ldots,0,1,y_1,0,...,0),\ldots,(0,\ldots,0,1,y_\ell)\Big)\Big\}$$

where the union is over all x, y with $\mathrm{Im}(y_i) = 0$ for all i and $\pm\mathrm{Im}(x_j) \geq 0$ depending on the choice of $+$ or $-$ in the j^{th} subscript of $Z_{\pm\pm\cdots\pm}$. These sets can be rewritten as

$$\begin{aligned} Z_{\pm\pm\cdots\pm} \;&=\; \big\{(z_2,...,z_{4n},w_1,...,w_{2\ell}) \,:\, \mathrm{Im}(z_2) \geq 0, \\ &\qquad \pm\,\mathrm{Im}(\overline{z_{2j-1}}z_{2j}) \geq 0 \;\;\forall j > 2, \;\;\text{and}\;\; \mathrm{Im}(\overline{w_{2i-1}}w_{2i}) = 0 \;\;\forall i\big\} \end{aligned}$$

The union of these, with orientations adjusted for signs, is the oriented semi-analytic set

$$\widetilde{Z} \;=\; \big\{(z,w) \in \mathbb{C}^{4n-1} \times \mathbb{C}^\ell \,:\, \mathrm{Im}(z_2) \geq 0 \;\;\text{and}\;\; \mathrm{Im}(\overline{w_{2i-1}}w_{2i}) = 0 \;\;\forall i\big\}$$

Thus, in this coordinate chart $\mathbb{C}^N/\mathbb{Z}_2$ our total cycle $\Gamma_{\bar{f}} \subset X^N$ is exactly the **reduced** image of the real analytic variety defined by the equations $\mathrm{Im}(\overline{w_{2i-1}}w_{2i}) = 0$, i.e.,

$$\Gamma_{\bar{f}} \cap (\mathbb{C}^N/\mathbb{Z}_2) \;=\; \tfrac{1}{2}\rho_*\big\{(z,w) \,:\, \mathrm{Im}(\overline{w_{2i-1}}w_{2i}) = 0 \;\;\forall i\big\}$$

where $\rho : \mathbb{C}^N \to \mathbb{C}^N/\mathbb{Z}_2$ is the projection.

Consider now the sphere

$$\widetilde{Y} \;=\; \Big\{(0,0,\ldots,0,1,it_0,1,it_1,\ldots,1,it_\ell) \,:\, t_i \in \mathbb{R} \;\;\forall i \;\;\text{and}\;\; \sum_i t_i^2 = 1\Big\}$$

and let $Y = \rho(\widetilde{Y}) \cong \mathbb{P}^\ell_{\mathbb{R}}$ be its reduced image in $\mathbb{C}^N/\mathbb{Z}_2 \subset X^N$. Note that Y misses the singular set $\rho(\mathbb{R}^N)$ and Y meets $\Gamma_{\bar{f}}$ in exactly one point, namely the conjugate pair corresponding to $t_1 = \cdots = t_\ell = 0$ and $t_0 = \pm 1$. One easily checks that this is a regular point of $\widetilde{Y}$. This completes the proof. $\square\,\square$

§A. Appendix: Splittings and Eilenberg-MacLane spaces.

A.1. Models for Eilenberg-MacLane spaces

In our discussion, the preferred model for the Eilenberg-MacLane space $K(\mathbb{Z}, n)$ is $\mathcal{Z}_0(S^n)_o$, the connected component of 0 in the topological abelian group $\mathcal{Z}_0(S^n)$. If N

is a finitely generated abelian group, then $\mathcal{Z}_0(S^n)_o \otimes_{\mathbb{Z}} N$ is our model for $K(N,n)$. Note that this model for $K(N,n)$ is a topological R-module, and in particular, this model for $K(\mathbb{Z}_p,n)$ is a *p-torsion group.*

Now, for each p fix a map $f_p : S^n \to S^n$ of degree p, and define

$$M(\mathbb{Z}_p,n) \stackrel{\text{def}}{=} D^{n+1} \cup_{f_p} S^n.$$

This is a Moore space satisfying

$$\widetilde{H}_j(M(\mathbb{Z}_p,n);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}_p & \text{if } j=k \\ 0 & \text{otherwise.} \end{cases}$$

It follows that we have yet another model for $K(\mathbb{Z}_p,n)$, namely, the *torsion free* abelian topological group $\mathcal{Z}_0(M(\mathbb{Z}_p,n))_o$. We need to establish a few properties of $M(\mathbb{Z}_p,n)$ and of $\mathcal{Z}_0(M(\mathbb{Z}_p,n))_o$.

Consider the canonical inclusion $\imath : S^n \hookrightarrow M(\mathbb{Z}_p,n)$, and let $F_p : D^{n+1} \to M(\mathbb{Z}_p,n)$ be the canonical map which induces the relative homeomorphism $F_p : (D^{n+1},\ S^n) \to (M(\mathbb{Z}_p,n),\ S^n)$ and satisfies $F_p\big|_{\partial D^{n+1}} = f_p$.

The following result is rather standard.

Lemma A.1. *Given a map $h : S^n \to Y$ such that $h \circ f_p$ is homotopic to zero, then there is an extension of h to $M(\mathbb{Z}_p,n)$. In other words, there is an $\overline{h} : M(\mathbb{Z}_p,n) \to Y$ such that $\overline{h} \circ \imath = h$. Furthermore, if $\pi_{n+1}(Y)=0$, then the extension is unique up to homotopy.*

Corollary A.2. *Let $h : S^n \to Y$ be as in Lemma A.1, and assume that Y is an abelian topological group.*

(1) *If h sends the base-point $x_\infty \in S^n$ to $0 \in Y$, then one has a commutative diagram*

$$\begin{array}{ccccc} S^n & \xrightarrow{\imath} & M(\mathbb{Z}_p,n) & \xrightarrow{\overline{h}} & Y \\ {\scriptstyle j_S}\big\downarrow & & \big\downarrow{\scriptstyle j_M} & & \big\downarrow = \\ \mathcal{Z}_0(S^n)_o & \xrightarrow[\imath_*]{} & \mathcal{Z}_0(M(\mathbb{Z}_p,n))_o & \xrightarrow[\overline{h}_*]{} & Y, \end{array}$$

where j_S and j_M are natural inclusions, and $\imath_*$ and $\overline{h}_*$ are the group homomorphisms induced by $\imath$ and $\overline{h}$, respectively.

(2) *If $\pi_{n+1}(Y) = 0$ then any continuous homomorphism $\phi : \mathcal{Z}_0(M(\mathbb{Z}_p,n))_o \to Y$, with the property $\phi \circ \imath_* \circ j_S = h$, is homotopic to $\overline{h}_*$ through continuous group homomorphisms with this property.*

Proof. The first assertion is a direct consequence of the fact that Y is an abelian topological group and from universal properties of the free abelian group on $M(\mathbb{Z}_p,n)$. To prove the second assertion, consider the map $\phi \circ j_M : M(\mathbb{Z}_p,n) \to Y$. Since $(\phi \circ j_M) \circ \imath = \phi \circ \imath_* \circ j_S = h$, one concludes from the proposition that $\phi \circ j_M$ is homotopic to $\overline{h}$ relative to i, and hence $\phi = (\phi \circ j_M)_*$ is homotopic to $\overline{h}_*$ through homomorphisms as claimed. $\square$

Corollary A.3. *Let $q : \mathcal{Z}_0(S^n)_o \to \mathcal{Z}_0(S^n)_o \otimes \mathbb{Z}_p$ denote the quotient map. Then there is a canonical homotopy equivalence $\Psi : \mathcal{Z}_0(M(\mathbb{Z}_p, n))_o \to \mathcal{Z}_0(S^n)_o \otimes \mathbb{Z}_p$ satisfying*

(1) *Ψ is a group homomorphism;*
(2) *$\Psi \circ \imath_* = q$.*

Furthermore, any Ψ' satisfying the above properties is homotopic to Ψ through such homomorphisms.

Proof. Just observe that the composition $(q \circ j_S) \circ f_p$ is homotopic to multiplication by p in the homotopy group $\pi_n(\mathcal{Z}_0(S^n)_o \otimes \mathbb{Z}_p)$, and hence $q \circ j_S$ satisfies the hypothesis of Corollary A.2. It is easy to see that the homomorphism resulting from Corollary A.2 induces an isomorphism of n-th homotopy groups and is therefore a homotopy equivalence. Since $q = (q \circ j_S)_*$, the result follows. $\square$

Corollary A.4. *Given an abelian topological group Y and a map $h : S^n \to Y$ such that $h \circ f_p \simeq 0$, there is a map*

$$H : \mathcal{Z}_0(S^n)_o \otimes \mathbb{Z}_p \to Y$$

satisfying $H \circ q \simeq h_*$. Furthermore, if $\pi_{n+1}(Y) = 0$, then any $\widetilde{H} : \mathcal{Z}_0(S^n)_o \otimes \mathbb{Z}_p \to Y$ with $\widetilde{H} \circ q \simeq h_*$ is homotopic to H through maps with this property.

Proof. Let $\Psi^{-1} : \mathcal{Z}_0(S^n)_o \otimes \mathbb{Z}_p \to \mathcal{Z}_0(M(\mathbb{Z}_p, n))_o$ be a homotopy inverse of the canonical Ψ defined in the previous Corollary, and let $\overline{h}_* : \mathcal{Z}_0(M(\mathbb{Z}_p, n)) \to Y$ be the homomorphism established in Corollary A.2. Define $H \stackrel{\text{def}}{=} \overline{h}_* \circ \Psi^{-1}$, and note that $H \circ q = \overline{h}_* \circ \Psi^{-1} \circ q \simeq \overline{h}_* \circ \imath_* = (\overline{h} \circ \imath)_* = h_*$, where the first equivalence follows from Corollary A.3. This is the desired H. To prove the last statement note that $\widetilde{H} \circ \Psi$ is homotopic to $\overline{h}_*$ through homomorphisms extending h. Hence, $\widetilde{H} = \widetilde{H} \circ \Psi \circ \Psi^{-1}$ is homotopic to $\overline{h}_* \circ \Psi^{-1} = H$. $\square$

A.2. Canonical splittings

It is a general theorem of J. Moore that any topological abelian group is homotopy equivalent to a product of Eilenberg-MacLane spaces. However, there are many inequivalent such splittings, and for the results in these papers and in [LM$_1$] one makes a canonical choice. For the particular examples of cycle groups that we study, the choice depends on the structure of $\mathbb{P}^n$ as a symmetric product of $\mathbb{P}^1$. However, in many cases the canonical splitting is determined *purely homotopy theoretically*. This is the main result of this Appendix. The existence of a theorem of this type was first pointed out to the authors by Eric Friedlander.

Throughout this discussion, the ring R will always be either $\mathbb{Z}$ or $\mathbb{Z}/n$, and we shall use the specific model $\mathcal{Z}_0(S^n)_o$ for the Eilenberg-MacLane space $K(\mathbb{Z}, n)$. More generally for any finitely generated module N over R, we shall take $K(N, n) = \mathcal{Z}_0(S^n)_o \otimes_{\mathbb{Z}} N$.

Let us fix a finitely generated R-module N and denote $K = K(N, n)$. One has an isomorphism

$$h_{K,R} : \pi_n(K) \otimes R = N \otimes R \xrightarrow{\cong} H_n(K; R),$$

obtained as the composition of isomorphisms $N\otimes R = \pi_n(K)\otimes R \xrightarrow{h_K\otimes I} H_n(K;\mathbb{Z})\otimes R \xrightarrow{\nu_K} H_n(K;R)$, where h_K is the Hurewicz map and ν_K is provided by the universal coefficients theorem for homology.

It follows from the universal coefficients theorem for cohomology that

$$\Psi_{K,R} \ : \ H^n(K;N) \to \mathrm{Hom}_R(H_n(K;R),N)$$

is an isomorphism, and the fundamental class $\iota_n \in H^n(K;N)$ is defined so that $\Psi_{K,R}(\iota_n)$ is the composition $H_n(K;R) \xrightarrow{h^{-1}_{K,R}} \pi_n(K)\otimes R = N\otimes R \xrightarrow{\mu_N} N$, where the latter map gives the R-module structure on N. Therefore,

$$\Psi_{K,R}(\iota_n) = \mu_N \circ h^{-1}_{K,R}. \tag{A.1}$$

We now examine these maps under the Dold-Thom theorem, which gives natural isomorphisms

$$d_{Y,R} \ : \ \pi_n(\mathcal{Z}_0(Y)\otimes R) \to H_n(Y;R) \tag{A.2}$$

for any CW-complex Y and for all n.

Since K is a topological abelian group, one has a topological homomorphism

$$t_K \ : \ \mathcal{Z}_0(K) \to K \tag{A.3}$$

such that the composition $\mathcal{Z}_0(K)\otimes R \xrightarrow{t_K\otimes I} K\otimes R \xrightarrow{\mu_K} K$ induces a left inverse

$$t_{K,R} \ : \ \mathcal{Z}_0(K)\otimes R \to K \tag{A.4}$$

to the natural inclusion

$$j_{K,R} \ : \ K \to \mathcal{Z}_0(K)\otimes R. \tag{A.5}$$

In the level of homotopy groups, this map fits into a commutative diagram

$$\begin{array}{ccc} N\otimes R = \pi_n(K)\otimes R & \xrightarrow[h_{K,R}]{\cong} & H_n(K;R) \\ {\scriptstyle \mu_N}\big\downarrow & & \big\uparrow{\scriptstyle \cong\ d_{K,R}} \\ N = \pi_n(K) & \xleftarrow[\pi_n(t_{K,R})]{} & \pi_n(\mathcal{Z}_0(K)\otimes R), \end{array}$$

which together with (A.1) implies that

$$\Psi_k(\iota_n) = \mu_n \circ h^{-1}_{K,R} = \pi_n(t_{K,R})\circ d^{-1}_{K,R}. \tag{A.6}$$

We now consider a map $f : Y \to K = K(N,n)$, representing a class $[f] \in H^n(Y,N)$. From the commutative diagram

$$\begin{array}{ccccc} H^n(Y;N) & \xrightarrow{\Psi_Y} & \mathrm{Hom}_R(H_n(Y;R),N) & \longrightarrow & 0 \\ {\scriptstyle H^n(f)}\uparrow & & \uparrow{\scriptstyle H_n(f)^*} & & \\ H^n(K;N) & \xrightarrow{\Psi_K} & \mathrm{Hom}_R(H_n(K;R),N) & \longrightarrow & 0 \end{array}$$

one concludes that

(A.7) $$\Psi_Y([f]) = \Psi_Y(H^n(f)(\iota_n)) = \Psi_K(\iota_n) \circ H_n(f)$$

and (A.6) gives

$$\Psi_Y([f]) = \pi_n(t_{K,R}) \circ d_{K,R}^{-1} \circ H_n(f).$$

Let

(A.8) $$\overline{f} : \mathcal{Z}_0(Y) \otimes R \to K$$

be the R-module homomorphism given by the composition $\mathcal{Z}_0(Y) \otimes R \xrightarrow{\mathcal{Z}_0(f)\otimes I} \mathcal{Z}_0(K) \otimes R \xrightarrow{t_{K,R}} K$. This map induces a commutative diagram

$$\begin{array}{ccccc} \pi_n(\mathcal{Z}_0(Y)\otimes R) & \xrightarrow{\pi_n(\mathcal{Z}_0(f)\otimes I)} & \pi_n(\mathcal{Z}_0(K)\otimes R) & \xrightarrow{\pi_n(t_{K,R})} & \pi_n(K) = N \\ {\scriptstyle d_{Y,K}}\downarrow & & {\scriptstyle d_{K,R}}\downarrow & & \| \\ H_n(K;R) & \xrightarrow[H_n(f)]{} & H_n(K;R) & \xrightarrow[\Psi_K(\iota_n)]{} & \pi_n(K) = N, \end{array}$$

where the left square commutes by the naturality of the Dold-Thom isomorphism, and the right square commutes by (A.6).

It follows that $\pi_n(\overline{f}) = \Psi_K(\iota_n) \circ H_n(f) \circ d_{Y,R}$, and by (A.7) one concludes that

(A.9) $$\pi_n(\overline{f}) = \Psi_Y([f]) \circ d_{Y,R}.$$

We now use the formulae above to prove the following result.

Theorem A.5. *Let Y be a connected finite complex and $R = \mathbb{Z}$ or $\mathbb{Z}/n$. Suppose that*

$$\Psi_Y : H^n(Y;\, H_n(Y;\, R)) \to Hom(H_n(Y;\, R),\, H_n(Y;\, R))$$

is an isomorphism for all n. Then there exists a homotopy equivalence

$$\mathcal{Z}_0(Y) \otimes_{\mathbb{Z}} R \xrightarrow[\cong]{\alpha} \prod_{k\geq 0} K(H_n(Y;\, R),\, n),$$

unique up to homotopy *with the property that:*

(i) α is a R-module homomorphism

(ii) The composition

$$Y \subset \mathcal{Z}_0(Y) \otimes R \xrightarrow{\alpha} \prod_{k\geq 0} K(H_n(Y;\, R),\, n)$$

classifies the identity element in $H^*(Y;\, H_*(Y;\, R)) \cong Hom(H_*(Y;\, R),\, H_*(Y;\, R))$.

Proof. Unless otherwise indicated all homology groups have coefficients in R.

We first prove existence. For each n there exists a map

$$f_n \ : \ Y \longrightarrow K(H_nY, n)$$

which classifies the identity element Id $\in$ Hom(H_nY, H_nY), i.e., $\Psi_Y([f_n]) = \text{Id}$. Now, define $F : \mathcal{Z}_0(Y) \otimes R \to \prod_{n\geq 0} K(H_n(Y), n)$ by $F = \prod_{n\geq 0} \overline{f}_n$, where $\overline{f}_n$ is defined as in (A.8), and note that F satisfies the two conditions of the theorem.

In the level of homotopy groups one has

$$\pi_n(F) = \pi_n(\overline{f}_n) : \pi_n(\mathcal{Z}_0(Y)) \to \pi_n(K(H_n(Y;R), n)) = H_n(Y;R),$$

and formula (A.9) shows that $\pi_n(f) = \Psi_Y([f_n]) \circ d_{Y,R} = \text{Id} \circ d_{Y,R} = d_{Y,R}$, the Dold-Thom isomorphism itself. It follows that F is a homotopy equivalence.

For uniqueness suppose we are given

$$Y \underset{j_{Y,R}}{\hookrightarrow} \mathcal{Z}_0(Y) \otimes G \ \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \ \prod_n K(H_nY, n)$$

where α and β are homotopy equivalences with properties (i) and (ii) above. These properties imply that $\alpha \circ j$ is homotopic to $\beta \circ j$. Since α and β are R-module homomorphsims and since $j_{Y,R}$ generates $\mathcal{Z}_0(Y) \otimes R$ as an R-module, this implies that $\alpha = \overline{\alpha \circ j_Y}$ is homotopic to $\beta = \overline{\beta \circ j_Y}$. $\square$

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