ALGEBRAIC CYCLES AND EQUIVARIANT COHOMOLOGY THEORIES

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1. Introduction

The main theme of this paper is that algebraic cycles provide interesting non-trivial invariants for finite groups, as well as new equivariant cohomology theories which answer natural questions in equivariant homotopy theory. Besides being quite computable, these theories carry Chern classes for representations and have deep relations with usual Borel cohomology theory. In fact their coefficients are simpler than standard group cohomology and have a geometric interpretation of independent interest. Among their main properties, we shall prove a full equivariant analogue of the Segal loop space conjecture proved in [3].

The genesis of the results is elementary and geometric. Nevertheless it directly yields a fully structured theory, graded over the representation ring and Mackey-functor-valued. Our construction is a natural extension of the standard construction of the classifying space for equivariant K-theory, and it fits beautifully into the theory of $\mathcal{I}_*$-functors, a machine which builds (equivariant) operad actions and infinite loop space structures. However, due to its elementary origins, the coefficients of the theory can be computed by geometric means. For example, using techniques of degeneration by $(\mathbb{C}^\times)^m$-actions we are able to compute π_0^G of our spectra when the group G is abelian. Its description relies on a remarkably simple and well-behaved graded ring $\mathbb{H}^*(G)$ characterized by the following properties:

(1) $\mathbb{H}^0(G) = \mathbb{Z}$ and $\mathbb{H}^1(G) \cong G$.
(2) $\mathbb{H}^*(G_1 \oplus G_2) = \mathbb{H}^*(G_1) \otimes \mathbb{H}^*(G_2)$
(3) If G is cyclic, then $\mathbb{H}^*(G) = H^{2*}(G;\mathbb{Z})$ (group cohomology with coefficients in the trivial G-module $\mathbb{Z}$).

For example, if G is abelian and $\mathfrak{z}_G$ denotes the first of our spectra, then the component ring $\pi_0^G(\mathfrak{z}_G)$ is exactly $\mathbb{H}^*(G)$ localized with respect to the multiplicative system engendered by the total Chern classes $1 + c_1(\rho) \in \{1\} \times \mathbb{H}^1(G)$ of the irreducible complex representations ρ of G. Some of the higher degree coefficients $\pi_k^H(\mathfrak{z}_G)$, $H < G$, may be computed by combining a general equivariant algebraic suspension theorem from [12] and the equivariant Dold-Thom theorem in [15].

Our basic constructions are modeled on the following "classical" one. Let G be a finite group. Recall that the classifying space for G-equivariant K-theory is a limit $B\mathcal{U}_G$ of Grassmann manifolds $\mathrm{Gr}(W) \equiv \mathrm{Gr}_w(W \oplus W)$ where $w = \dim(W)$ and where W ranges over a family of linear G-spaces for which the multiplicity of each irreducible representation tends to infinity. This limit carries an H-space structure induced by the direct sum $\oplus : \mathrm{Gr}(W) \times \mathrm{Gr}(W') \to \mathrm{Gr}(W \oplus W')$. The internal coherence of this multiplication is succinctly captured in the fact that $(\mathrm{Gr}(\bullet), \oplus)$ is an equivariant $\mathcal{I}_*$-functor. Hence, by general theory [4], the $\oplus$-multiplication enhances to an equivariant infinite loop space structure, and thereby $B\mathcal{U}_G$ becomes the zero'th space in a G-spectrum $\mathfrak{ku}_G$, which gives connective equivariant K-theory.

We mimic this construction by considering an analogous family $\mathcal{Z}^w(\mathbb{P}(W \oplus W))$ of algebraic cycles of codimension w on $\mathbb{P}(W \oplus W)$. Each $\mathcal{Z}(W)$ is a *topological* abelian group, and the construction comes equipped with a biadditive "external" pairing on these groups defined by the algebraic join $\sharp_{\mathbb{C}}$ of cycles. Taking the analogous limit over W defines the basic space of our theory $\mathcal{Z}(\mathcal{U}_G)$, the group of "algebraic cycles on $\mathbb{P}^\infty$". This is a topological abelian group with a biadditive pairing induced by $\sharp_{\mathbb{C}}$. Furthermore, at finite levels $(\mathcal{Z}(\bullet), \sharp_{\mathbb{C}})$ is an $\mathcal{I}_*$-monoid. Our first main result, a combination of Propositions 2.4.2, 3.1.2 and 3.3.3 implies that

$$\mathcal{Z}(\mathcal{U}_G) \text{ is a } G\text{-group complete equivariant } E_\infty\text{-ring space.}$$

In particular, there is a G-equivalence $\imath : \mathcal{Z}(\mathcal{U}_G) \to \mathfrak{z}_G(0)$ with the degree-zero space of an equivariant E_∞-ring spectrum $\mathfrak{z}_G$. These spectra $\mathfrak{z}_G$ form a *coherent family* over the category of finite groups (see §2.3), and each $\mathfrak{z}_G$ admits an *augmentation* $\delta : \mathfrak{z}_G \to \mathbb{Z}$ to the "constant" ring spectrum $\mathbb{Z}$.

The subset $\mathcal{Z}(\mathcal{U}_G)_1 = \epsilon^{-1}\delta^{-1}(1)$ of cycles of degree one is closed under the (homotopy commutative) join multiplication, and the same machinery tells us that there is a second "multiplicative" G-spectrum $\mathfrak{M}_G$ and a map

$$\epsilon_1 : \mathcal{Z}(\mathcal{U}_G)_1 \longrightarrow \mathfrak{M}_G(0)$$

which is a G-**group completion** with respect to join pairing. For the trivial group $G = \{1\}$ this is actually an equivalence. However, the calculation of coefficients alluded to above shows that $\mathcal{Z}(\mathcal{U}_G)_1$ is not in general a *group-like space over* G, that is, $\pi_0(\mathcal{Z}(\mathcal{U}_G)_1^H)$ is not a group under $\sharp_{\mathbb{C}*}$ for non-trivial $H < G$. The completion $\mathfrak{M}_G(0)$ is a connected space for which this is true. In a similar fashion to $\mathfrak{z}_G$, the spectra $\mathfrak{M}_G$, for various G's, form a coherent family; cf. Theorem 4.1.4. As a consequence, both $\mathfrak{z}_G$ and $\mathfrak{M}_G$ are *split* spectra, in the sense of [14, I.8]; cf. Corollary 2.3.3.

We observe now that considering linear subspaces to be algebraic cycles of degree one gives an equivariant inclusion $\mathrm{Gr}(W) \subset \mathcal{Z}(W)$ for any finite-dimensional representation W of G. Furthermore on linear subspaces the join pairing is simply the direct sum. Hence there is a G-equivariant map

$$c_G : B\mathcal{U}_G = \varinjlim_W \mathrm{Gr}(W) \;\to\; \varinjlim_W \mathcal{Z}(W) = \mathcal{Z}(\mathcal{U}_G) \tag{1.1}$$

taking $\oplus$ to $\sharp_{\mathbb{C}}$ and having image in $\mathcal{Z}(\mathcal{U}_G)_1$. In fact, the inclusions above constitute a natural transformation $(\mathrm{Gr}(\bullet), \oplus) \to (\mathcal{Z}(\bullet), \sharp_{\mathbb{C}})$ of $\mathcal{I}_*$-functors; cf. Proposition 3.2.1. From this one can prove that the map $B\mathcal{U}_G \to \mathcal{Z}(\mathcal{U}_G)_1$ enhances to a transformation of G-spectra

$$c_G : \mathfrak{ku}_G \longrightarrow \mathfrak{M}_G. \tag{1.2}$$

Suppose we take G to be the trivial group. Then it was proved in [11] that there is a canonical homotopy equivalence

$$\mathcal{Z}(W) \cong K(\mathbb{Z},0) \times K(\mathbb{Z},2) \times K(\mathbb{Z},4) \times \cdots \times K(\mathbb{Z},2w). \tag{1.3}$$

Furthermore, in [13] it was shown that under (1.3) the join pairing $\sharp_{\mathbb{C}} : \mathcal{Z}(W) \times \mathcal{Z}(W') \to \mathcal{Z}(W \oplus W')$ classifies the cup product, and the inclusion $\mathrm{Gr}(W) \subset \mathcal{Z}(W)$ classifies the total Chern class of the tautological w-plane bundle over $\mathrm{Gr}(W)$. In particular, if one drops the group in (1.1) by forgetting the G-action, then one finds the transformation of classifying spaces

$$c : BU \longrightarrow \{1\} \times \prod_{k>0} K(\mathbb{Z}, 2k) \tag{1.4}$$

corresponding to the total Chern class. The compatibility of this map with $\sharp_{\mathbb{C}}$ corresponds to the standard formula $c(E \oplus E') = c(E)c(E')$ for vector bundles over a compact space. In 1975 G. Segal [23] conjectured that there exists an infinite loop space structure on $\{1\} \times \prod_{k>0} K(\mathbb{Z}, 2k)$ such that (1.4) becomes an infinite loop map. The assertion (1.2) in the case where $G = \{1\}$ constitutes a proof of this conjecture which was established in [3]. The general assertion (1.2) is the **full equivariant analogue of Segal's conjecture**; see Definition 4.2.1.

It is natural to ask if there are analogues of the results above for standard Borel cohomology theory. In a second set of results we establish such analogues. Indeed given any G-spectrum $\mathbf{k}_G$ there are an **associated Borel spectrum** $\mathbf{bk}_G$ and a map of G-spectra

$$\epsilon : \mathbf{k}_G \longrightarrow \mathbf{bk}_G$$

which is a non-equivariant homotopy equivalence; cf. [8]. The spectrum $\mathbf{bk}_G$ is simply defined by setting $\mathbf{bk}_G = F(EG_+, \mathbf{k}_G)$ where EG denotes as usual a contractible CW-complex on which G acts freely, and where $F(X, Y)$ denotes the base-point preserving continuous maps from X to Y. The transformation ϵ is induced from the map $EG \to \mathrm{pt}$.

In §4 we compute the Borel spectra $\mathbf{b}\mathfrak{Z}_G$ and $\mathbf{b}\mathfrak{M}_G$ associated to our "algebraic cycle" spectra $\mathfrak{Z}_G$ and $\mathfrak{M}_G$ above. It turns out that for any compact G-space X there are natural isomorphisms

$$[X, \mathbf{b}\mathfrak{Z}_G(0)]_G \cong H^{2*}_G(X; \mathbb{Z})_{\mathrm{Borel}} \quad \text{and} \quad [X, \mathbf{b}\mathfrak{M}_G(0)]_G \cong \{1\} \times \prod_{k>0} H^{2k}_G(X; \mathbb{Z})_{\mathrm{Borel}};$$

see Theorems 4.1.5 and 4.1.6 and their corollaries.

A combination of Theorems 4.1.6 and 4.2.2 yields the following statement:

Theorem A. *Let G be a finite group. The map of G-spectra*

$$c^b_G : \mathfrak{ku}_G \longrightarrow \mathbf{b}\mathfrak{M}_G$$

obtained by composing ϵ with (1.2) above classifies the total Chern class

$$c : K_G(X) \longrightarrow \{1\} \times \prod_{k>0} H^{2k}_G(X; \mathbb{Z})$$

in Borel cohomology. The multiplications on the zero'th level, which are respected by this map of spectra, correspond to the formula $c(E \oplus E') = c(E)c(E')$ for equivariant bundles over a compact G-space X.

Said in another way, **the multiplication of units in Borel cohomology, considered as an H-space structure on the classifying space, enhances to to an equivariant infinite loop space structure so that the total Chern class from $\mathfrak{ku}_G$ becomes an**

infinite loop map in the category of G-spaces. This is precisely the Segal loop space conjecture for Borel cohomology.

We also obtain analogous results for equivariant KO-theory and Stiefel-Whitney classes, by using real algebraic cycles and the work of T. K. Lam [10]; cf. §4.4. We construct spectra $\mathfrak{Z}_{\mathbb{R},G}$ and $\mathfrak{M}_{\mathbb{R},G}$, together with maps of spectra $w_G : \mathfrak{ko}_G \rightarrow \mathfrak{M}_{\mathbb{R},G}$, where $\mathfrak{ko}_G$ is the spectrum of equivariant connective KO-theory. A similar combination of results yields the corresponding statement:

Theorem B. *Let G be a finite group. The map of G-spectra*

$$w_G^b : \mathfrak{ko}_G \rightarrow \mathbf{b}\mathfrak{M}_{\mathbb{R},G}$$

obtained by composing ϵ with (1.2) *above classifies the total Stiefel-Whitney class*

$$w : KO_G(X) \rightarrow \{1\} \times \prod_{k>0} H_G^k(X;\, \mathbb{Z}/2\mathbb{Z})$$

in Borel cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients. The multiplications on the zero'th level, which are respected by this map of spectra, correspond to the formula $w(E \oplus E') = w(E)w(E')$ for equivariant bundles over a compact G-space X.

There are several consequences of the naturality of these spectra and the maps relating them, which remain to be fully explored. For example, these maps commute with the transfers associated to equivariant bundles, for the various theories. In particular, one has multiplicative transfers for even group cohomology which naturally compute Chern classes of induced representations; one computes equivariant Chern classes for the push-forward of bundles under finite equivariant covering spaces; naturality properties for the composition of coverings follow trivially from the nature of the constructions, etc. See §4.3 for a brief discussion.

With an aim at algebro-geometric applications, we end the paper with a generalization of these constructions which associates to every complex algebraic variety X with a finite group of automorphisms G, the **morphic spectra** $\mathfrak{Z}_{X,G}$ and $\mathfrak{M}_{X,G}$. These spectra reflect both the algebraic and topological properties of X and provide an extension of the notion of *morphic cohomology* introduced and studied in [6]. In short, we obtain a natural transformation from the category of algebraic varieties over $\mathbb{C}$ to the category of spectra, respecting actions of finite groups of automorphisms and endowed with a natural theory of Chern classes. The spectra studied in this paper are obtained from these more general objects by taking $X = \mathrm{pt}$. See §6.1.

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2. General Theory

In this chapter we recall the fundamental concepts of equivariant cohomology theories and their associated Borel counterparts. This will include the equivariant theory of infinite loop spaces, with emphasis on a "recognition principle" for detecting such spaces. The main sources for background material are [14], [4] and [8].

2.1. **Equivariant Cohomology Theories and G-Spectra.** A **universe** $\mathcal{U}$ for a compact Lie group G is a unitary (or orthogonal) representation containing countably many copies of each of its irreducible subrepresentations. We fix, for every compact Lie group G, a **complete G-universe** $\mathcal{U}_G$, in other words, a universe containing all irreducible representations of G, including the trivial one. The universes $\mathcal{U}_G$ will parameterize all representations we deal with, and hence, whenever we refer to a **G-module** V or an **indexing G-space** V, we will be implicitly working with a finite dimensional unitary subrepresentation of $\mathcal{U}_G$.

Definition 2.1.1. We assume that all spaces are based with G-fixed base point, and use the symbol X_+ to denote the disjoint union $X \amalg \{*\}$ of X with a G-fixed base point. The function space of based maps between G-spaces X and Y is denoted by $F(X,Y)$, and is considered as a G-space under the standard action by "conjugation". For a G-space X and G-module V, the V-th suspension of X is the G-space $\Sigma^V X{:=}X \wedge S^V$, where S^V denotes the one-point compactification of V with base point at ∞, and the V-th loop space of X is $\Omega^V X{:=}F(S^V, X)$, both taken with their natural G-actions.

In this equivariant context, given an appropriate representation ring $\mathcal{R}$, e.g. $\mathcal{R} = RO(G)$ or $\mathcal{R} = R(G)$, we are going to construct natural $\mathcal{R}$-graded cohomology theories. These are equivariant cohomology theories $\mathfrak{h}^*_G$ satisfying well-known axioms, including contravariance and invariance under equivariant maps and homotopies; exactness for equivariant cofiber sequences etc. The reader is referred to [24, Defn. 6.8] for a full description of these axioms, although we would like to emphasize that the main additional feature of these theories, as opposed to the non-equivariant generalized cohomology theories, is the existence of suspension isomorphisms associated to arbitrary G-modules. More precisely, the theories $\mathfrak{h}^*_G$ considered here come equipped with natural isomorphisms $\sigma_V : \mathfrak{h}^\alpha_G(X) \to \mathfrak{h}^{\alpha+V}_G(\Sigma^V X)$ for each $\alpha \in \mathcal{R}$ and G-module V.

Equivariant theories have been the focus of intense investigation in recent years, and have proven to be very useful in dealing with both equivariant and non-equivariant questions in stable homotopy theory. The monographs [14] and [8] contain a myriad of examples as well as an extensive bibliography. For further material, the reader may also consult [24] and [9].

Definition 2.1.2. Let $\mathcal{U}$ be a universe for G, not necessarily *complete*.

a: A **G-prespectrum** $\mathbf{k}_G$ indexed on $\mathcal{U}$, consists of a collection $\{\mathbf{k}_G(V) \mid V \subset \mathcal{U}\}$ of based G-spaces, together with a transitive system of G-maps $\sigma_V^W : \mathbf{k}_G(V) \to \Omega^{W-V}\mathbf{k}_G(W)$, where V is a G-submodule of W and $W-V$ denotes the orthogonal complement of V in W; cf. [14, Defn. 2.1]. A **morphism** of G-prespectra $f : \mathbf{k}_G \to \mathbf{k}'_G$ is a collection of G-maps $f_V : \mathbf{k}_G(V) \to \mathbf{k}'_G(V)$ which is strictly compatible with the corresponding structural maps.

The collection of all G-prespectra indexed on $\mathcal{U}$ forms a category which we denote by $G\mathcal{PU}$, in which the set of morphims from $\mathbf{k}_G$ to $\mathbf{k}'_G$ is denoted by $G\mathcal{PU}(\mathbf{k}_G, \mathbf{k}'_G)$.

b: A G-prespectrum $\mathbf{k}_G$, indexed on $\mathcal{U}$ is called a G-spectrum whenever the structural maps σ_V^W are G-homeomorphisms. The category of G-spectra indexed on the universe $\mathcal{U}$, denoted by $G\mathcal{SU}$, forms a full subcategory of the category of G-prespectra.

c: Given a G-spectrum $\mathbf{k}_G$ and a G-space X, the collection $F(X, \mathbf{k}_G(V))$ is again a G-spectrum under the evident natural maps. In particular, one can define for every G-spectrum $\mathbf{k}_G$ and G-module V, the V-th loop spectrum $\Omega^V \mathbf{k}_G := F(S^V, \mathbf{k}_G)$ of $\mathbf{k}_G$.

Example 2.1.3. Let $\mathbf{k}_G$ be a G-spectrum and let EG denote a contractible $\mathcal{CW}$-complex on which G acts freely. The **associated Borel spectrum $\mathbf{bk}_G$** to $\mathbf{k}_G$ is the G-spectrum $F(EG_+, \mathbf{k}_G)$. We shall see afterwards that this notion represents the adequate notion of Borel cohomology associated to $\mathbf{k}_G$. Notice that the projection $EG \to \{pt\}$ induces a natural map of G-spectra $\epsilon : \mathbf{k}_G \to \mathbf{bk}_G$.

A G-prespectrum is automatically a G-spectrum, and hence there is a "forgetful" functor $\ell : G\mathcal{SU} \to G\mathcal{PU}$. This functor admits a left adjoint L which solves the problem that the category of G-spectra is not closed under basic space-level constructions, such as taking smash products, wedges, cofibers, etc. This is the content of the following fundamental result; cf. [14, Thm I.2.2].

Theorem 2.1.4. *There is a left adjoint $L : G\mathcal{PU} \to G\mathcal{SU}$ to the forgetful inclusion functor $\ell : G\mathcal{SU} \to G\mathcal{PU}$. That is, $G\mathcal{PU}(\mathbf{k}_G, \ell\mathbf{h}_G) \cong G\mathcal{SU}(L\mathbf{k}_G, \mathbf{h}_G)$ for $\mathbf{k}_G \in G\mathcal{PU}$ and $\mathbf{h}_G \in G\mathcal{SU}$. Let $\eta : \mathbf{k}_G \to \ell L\mathbf{k}_G$ and $\epsilon : L\ell\mathbf{h}_G \to \mathbf{h}_G$ be the unit and counit of the adjunction. Then ϵ is an isomorphism for each G-spectrum $\mathbf{h}_G$, hence η is an isomorphism if $\mathbf{k}_G = \ell\mathbf{h}_G$.*

The functor L allows one to perform constructions with spectra whose properties are derived easily from the fact that L is a left adjoint. For example, given a G-space X and $\mathbf{k}_G \in G\mathcal{PU}$ one defines a G-prespectrum $X \wedge \mathbf{k}_G$ by sending V to $X \wedge \mathbf{k}_G(V)$. If $\mathbf{k}_G$ is a spectrum, then the G-spectrum $X \wedge \mathbf{k}_G \in G\mathcal{SU}$ is defined as $L(X \wedge \ell\mathbf{k}_G)$. Similar procedures allow one to define cones, suspensions and homotopies in the category of spectra. In particular, the **suspension spectrum** $\Sigma^\infty X$ of a G-space X is defined as $L(S^\infty X)$, where $S^\infty X$ is the prespectrum $V \mapsto X \wedge S^V$.

Definition 2.1.5. Given a G-spectrum $\mathbf{h}_G \in G\mathcal{SU}_G$, one defines an $R(G)$-graded **cohomology theory $\mathbf{h}_G^*$** on G-spectra by assigning to $\mathbf{k}_G \in G\mathcal{SU}_G$ the groups

$$\mathbf{h}_G^\alpha(\mathbf{k}_G) := \{\Sigma^W \mathbf{k}_G, \Sigma^V \mathbf{h}_G\}_G,$$

where one writes the virtual representation $\alpha \in R(G)$ as a formal difference of two G-modules $\alpha = V - W$. The symbol $\{\Sigma^W \mathbf{k}_G, \Sigma^V \mathbf{h}_G\}_G$ denotes equivariant homotopy classes of maps of G-spectra, and $\Sigma^W \mathbf{k}_G$, $\Sigma^V \mathbf{h}_G$ denote the appropriate suspension of the spectra $\mathbf{k}_G$ and $\mathbf{k}'_G$, respectively. In particular, the V-th **cohomology group $\mathbf{h}_G^V(X)$** of a G-space X of the homotopy type of a G-$\mathcal{CW}$-complex is retrieved by $\mathbf{h}_G^V(\Sigma^\infty X_+) = [X, \mathbf{h}_G(V)]_G$, where $[X, \mathbf{h}_G(V)]_G$ denotes equivariant homotopy classes of maps of G-spaces.

Changes of Universe

Given a G-linear isometric embedding $\jmath : \mathcal{U} \to \mathcal{V}$ between two G-universes $\mathcal{U}$ and $\mathcal{V}$, one can define a pair of adjoint **change of universe** functors $\jmath^* : G\mathcal{SV} \to G\mathcal{SU}$ and $\jmath_* : G\mathcal{SU} \to G\mathcal{SV}$ as follows.

For $\mathbf{k}_G \in G\mathcal{SV}$, one defines $\jmath^*\mathbf{k}_G \in G\mathcal{SU}$ by $\jmath^*\mathbf{k}_G(V) = \mathbf{k}_G(\jmath(V))$, where the adjoints of the structural maps are given by the compositions

$$\Sigma^{W-V}\jmath^*\mathbf{k}_G(V) = \mathbf{k}_G(\jmath(V)) \wedge S^{W-V} \xrightarrow{1\wedge\jmath} \mathbf{k}_G(\jmath(V)) \wedge S^{\jmath(W)-\jmath(V)} \xrightarrow{\tilde{\sigma}} \mathbf{k}_G(\jmath(W)) = \jmath^*\mathbf{k}_G(W).$$

It is evident that this construction takes spectra to spectra.

We first define the covariant functor $\jmath_*$ in the level of prespectra. Given a G-prespectrum $\mathbf{k}_G$ indexed on $\mathcal{U}$, the assignment $V \mapsto \mathbf{k}_G(\jmath^{-1}V) \wedge S^{V-\jmath(\jmath^{-1}V)}$ naturally defines a G-prespectrum $\jmath_*\mathbf{k} \in G\mathcal{PV}$. In the level of spectra, one defines $\jmath_*(E_G){:=}L(\jmath_*(\ell E_G))$, and it is easy to see that $\jmath_*$ is a left adjoint of $\jmath^*$; cf. [14, I.1.2].

Naïve G-spectra

An important instance of change of universes is given by the G-linear isometry $\imath : \mathcal{U}^G \to \mathcal{U}$, where $\mathcal{U}^G$ is the G-fixed point space of $\mathcal{U}$, considered as a "trivial" G-universe identifiable with $\mathbb{R}^\infty$. The G-spectra indexed on $\mathcal{U}^G$ are simply ordinary spectra with actions of G on their structural spaces. These spectra are called *naïve* G-spectra, and the usual, nonequivariant spectra can be considered as naïve G-spectra with trivial G-action. Following [8], we denote the category of naïve G spectra by $G\mathcal{SU}^G$, and recall the following result from [14, II.1.8], which relates the passage from naïve to genuine G-spectra and vice-versa, given by the change of universe functors induced by $\imath : \mathcal{U}^G \to \mathcal{U}$.

Lemma 2.1.6. *For $\mathbf{k}_G \in G\mathcal{SU}^G$, the unit map $\eta : \mathbf{k}_G \to \imath^*\imath_*\mathbf{k}_G$ of the $(\imath^*, \imath_*)$ adjunction is a nonequivariant equivalence. For $\mathbf{k}_G \in G\mathcal{SU}$, the counit map $\epsilon : \imath_*\imath^*\mathbf{k}_G \to \mathbf{k}_G$ is a nonequivariant equivalence.*

Split Spectra

Since the indexing sets for naïve G-spectra are trivial G-spaces, one can naturally associate to $\mathbf{k}_G \in G\mathcal{SU}^G$ its fixed point (nonequivariant) spectrum $(\mathbf{k}_G)^G \in \mathcal{SU}^G$, which sends $V \subset \mathcal{U}^G$ to $(\mathbf{k}_G(V))^G$. If $\mathbf{k}_G$ is an actual G-spectrum, one defines $(\mathbf{k}_G)^G = (\imath^*\mathbf{k}_G)^G$.

Definition 2.1.7. A naïve G-spectrum is said to be **split** if there is a map of spectra $\xi : \mathbf{k} \to (\mathbf{k}_G)^G$, from the underlying nonequivariant spectrum $\mathbf{k}$ of $\mathbf{k}_G$ to its fixed point spectrum $(\mathbf{k}_G)^G$, whose composition with the inclusion of $(\mathbf{k}_G)^G$ into $\mathbf{k}$ is homotopic to the identity map. A G-spectrum $\mathbf{k}_G \in G\mathcal{SU}$ is called split if $\imath^*\mathbf{k}_G \in G\mathcal{SU}^G$ is split.

Split spectra play an important role in both equivariant and nonequivariant contexts, and among their pleasant characteristics is the following result; cf. [14, I.8.4] or [9].

Proposition 2.1.8. *Let $\mathbf{k}_G$ be a split G-spectrum, and let X be a space on which G acts freely. Then $\mathbf{k}^*_G(X) \cong \mathbf{k}^*(X/G)$, where $\mathbf{k}^*$ is the nonequivariant cohomology theory associated with the underlying nonequivariant spectrum $\mathbf{k}$ of $\mathbf{k}_G$.*

Corollary 2.1.9. *If $\mathbf{k}_G$ is a split G-spectrum, and $\mathbf{b}(\mathbf{k}_G)^*$ denotes the Borel cohomology associated to $\mathbf{k}_G$, then for any G-space X one has*

$$\mathbf{b}(\mathbf{k}_G)^*(X) \cong \mathbf{k}^*(EG \times_G X).$$

Remark 2.1.10. This corollary justifies the assertion that $\mathbf{b}(\mathbf{k}_G)^*$ is the adequate Borel cohomology associated to $\mathbf{k}_G^*$.

2.2. **Equivariant infinite loop spaces.** A good source of non-equivariant (connective) spectra is the classical theory of infinite loop spaces [1], [18]. Unfortunately, to our knowledge, in the equivariant context such machines are only well-understood in the case of finite groups. Therefore, despite the fact that our constructions hold for arbitrary compact Lie groups, we need to restrict ourselves to the case of finite groups in order to apply the existing equivariant infinite loop space machines. Throughout the rest of this section we work with a fixed finite group G and its universe $\mathcal{U} = \mathcal{U}_G$.

Definition 2.2.1. An **equivariant infinite loop space**, indexed by $\mathcal{U}$, is a G-space X such that for every indexing space $V \subset \mathcal{U}$ there is a G-space $Y(V)$ together with a homeomorphism $X \cong \Omega^V Y(V)$. Furthermore, if for a given submodule $W \subset V$ one denotes by $V - W$ the orthogonal complement of W in V, then there are compatible G-homeomorphisms $Y(W) \cong \Omega^{V-W} Y(V)$. It is evident that an infinite loop space X is the zero-th space $\mathbb{X}(0)$ of an equivariant G-spectrum $\mathbb{X}$.

Infinite loop space theories come equipped with a "recognition principle" which tells when a space is an infinite loop space or, more generally, with an "infinite loop machine" which produces infinite loop spaces out of spaces with a certain structure. The machine that we use requires the notion of a G-operad acting on a G-space X. We refer the reader to [14,] or [4] for the precise definition of such object.

Presently we only need to introduce the **linear isometries operad** $\mathcal{L}(\mathcal{U})$ associated to a G-universe $\mathcal{U}$, which consists of the collection of spaces $\{\mathcal{L}(\mathcal{U})_k \mid k \in \mathbb{Z}_+\}$ and structural maps $\gamma : \mathcal{L}(\mathcal{U})_k \times \mathcal{L}(\mathcal{U})_{j_1} \cdots \times \cdots \mathcal{L}(\mathcal{U})_{j_k} \to \mathcal{L}(\mathcal{U})_{j_1+\cdots+j_k}$, where $\mathcal{L}(\mathcal{U})_k$ is the space $\mathcal{I}(\mathcal{U}^{\oplus k}, \mathcal{U})$ of linear isometries from $\mathcal{U}^{\oplus k}$ into $\mathcal{U}$ and γ is the natural map given by composition $\gamma : (f; g_1, \ldots, g_k) \mapsto f \circ (g_1 \oplus \cdots \oplus g_k)$. The spaces $\mathcal{L}(\mathcal{U})_k$ admit a natural action of $G \times \mathfrak{S}_k$, where $\mathfrak{S}_k$ is the symmetric group on k letters, and the structural maps satisfy suitable equivariant compatibility conditions. Furthermore, $\mathcal{L}(\mathcal{U})_k$ is a G-$\mathcal{CW}$-complex on which $\mathfrak{S}_k$ acts freely and, if $\mathcal{U}$ is complete universe, the fixed point set $\mathcal{L}(\mathcal{U})_k^\Lambda$ is contractible, for each subgroup $\Lambda \subset G \times \mathfrak{S}_k$ such that $\Lambda \cap \mathfrak{S}_k$ is trivial. In other words, $\mathcal{L}(\mathcal{U})_k$ is a universal principal $(G, \mathfrak{S}_k)$-bundle in the sense of [24].

One says that a G-space X is an *algebra over an operad* $\mathcal{L}$, or that $\mathcal{L}$ acts on X if there is a collection of $G \times \mathfrak{S}_k$-maps $\lambda : \mathcal{L}_k \times (X \times \ldots \times X) \to X$ satisfying appropriate compatibility relations with the structural maps of the operad.

The relevance of the notion of $\mathcal{L}(\mathcal{U})$-spaces for our work lies in the following recognition principle, due to Hauschild, May and Waner, which generalizes earlier work of May [18], [19].

Theorem 2.2.2 ([4, Thm.1]). *Let X be a group-like based G-space of the homotopy type of a G-$\mathcal{CW}$-complex. If X is algebra over $\mathcal{L}(\mathcal{U})$, where $\mathcal{U}$ is a complete G-universe, then X is G-homotopy equivalent to an equivariant infinite loop space.*

Now recall that a Hopf space[1] map $M \to \tilde{M}$ is a called a **group completion** (in the homotopy-theoretic sense) if for any coefficient ring R, the induced homomorphism of Pontrjagin rings $H_*(M,R) \to H_*(\tilde{M},R)$ is the localization of $H_*(M,R)$ with respect to the multiplicative subset generated by $\pi_0(M)$.

In the category of G-spaces, a G-map $X \to Y$ between two Hopf G-spaces is a **G-group completion** if each induced map $X^H \to Y^H$ is a (non-equivariant) group completion, for all $H \leq G$.

In this framework, the proof of the previous theorem actually yields the following result, where we assume that X is any space of the homotopy type of a G-$\mathcal{CW}$-complex.

Theorem 2.2.3. *Let X be a $\mathcal{L}(\mathcal{U})$-space, where $\mathcal{U}$ is a complete G-universe. Then there is an equivariant infinite loop space $\mathbb{X}(0)$, and a G-map of $\mathcal{L}(\mathcal{U})$-spaces $\imath : X \to \mathbb{X}(\not\vdash)$ which is a G-group completion.*

2.3. **Coherence.** In this section we shall analyze $\mathcal{L}(\mathcal{U})$-spaces which have particularly nice properties with respect to the subgroups of G. Our main result asserts that such spaces yield spectra which are *split*, and whose Borel counterparts are therefore "classical" by (2.1.9).

In the following exposition we emphasize the dependence of $\mathcal{U}$ on G by writing $\mathcal{U}_G$ in place of $\mathcal{U}$.

Let $\mathcal{G}$ denote the category whose objects are the subgroups of G, and whose morphisms are the inclusions $\alpha_{H',H} : H \hookrightarrow H'$, the projections $\beta_H : H \to \{e\}$, and their compositions. For each $H \in \mathcal{G}$ one can take $\mathcal{U}_H = \alpha^*_{G,H}(\mathcal{U}_G)$. To each inclusion $\alpha = \alpha_{H',H}$ and each projection $\beta = \beta_H$ in $\mathcal{G}$ we associate the H-isometries

$$j_\alpha : \alpha^*\mathcal{U}_{H'} \to \mathcal{U}_H \text{ and } j_\beta : \beta^*\mathcal{U}_e \to \mathcal{U}_H \tag{2.1}$$

by letting j_α be the identity map, and by defining j_β to be the composition $\mathcal{U}_e \xrightarrow{\cong} (\mathcal{U}_H)^H \hookrightarrow \mathcal{U}_H$. These homomorphisms and isometries induce **change of group functors** among spectra, as follows; cf. [14].

A group homomorphism $\alpha : H \to G$ naturally defines a functor $\alpha^* : G\mathcal{S}\mathcal{U}_G \to H\mathcal{S}(\alpha^*\mathcal{U}_G)$ which considers an indexing G-space $V \subset \mathcal{U}_G$ as an indexing H-space via α. For α in our category $\mathcal{G}$ we now define the first **change of group** functor $\alpha^\sharp : G\mathcal{S}\mathcal{U}_G \to H\mathcal{S}\mathcal{U}_H$ to be the composition $G\mathcal{S}\mathcal{U}_G \xrightarrow{\alpha^*} H\mathcal{S}(\alpha^*\mathcal{U}_G) \xrightarrow{j_{\alpha *}} H\mathcal{S}\mathcal{U}_H$, where $j_{\alpha *}$ is induced by the change of universe $j_\alpha : \alpha^*\mathcal{U}_G \to \mathcal{U}_H$ defined above.

The functor $\alpha^\sharp$ has a natural "right homotopy adjoint" $\alpha_\sharp : H\mathcal{S}\mathcal{U}_H \to G\mathcal{S}\mathcal{U}_G$ with $\{\alpha^\sharp \mathbf{h}_G, \mathbf{k}_H\}_H \cong \{\mathbf{h}_G, \alpha_\sharp \mathbf{k}_H\}_G$, defined as follows. Given an H-prespectrum $\mathbf{t}_H \in H\mathcal{P}(\alpha^*\mathcal{U}_G)$, one can define a G-prespectrum $F_\alpha[G, \mathbf{t}_H)$ which sends $V \in \mathcal{U}_G$ to the function space

[1]An algebra over an operad (in the category of spaces) is automatically a Hopf space.

$F_\alpha(G_+, \mathbf{t}_H(V))$ of left H-maps $G_+ \longrightarrow \mathbf{t}_H(V)$ with left action on $F_\alpha(G, \mathbf{t}_H(V))$ induced by the right action of G on itself. At the level of spectra, one then defines a functor $\alpha_* : H\mathcal{S}(\alpha^*\mathcal{U}_G) \longrightarrow G\mathcal{S}\mathcal{U}_G$ as $\alpha_*(\mathbf{k}) = L(F_\alpha[G, \ell\mathbf{k}))$. One finally introduces the desired change of group functor $\alpha_\sharp : H\mathcal{S}\mathcal{U}_H \longrightarrow G\mathcal{S}\mathcal{U}_G$ as the composition $H\mathcal{S}\mathcal{U}_H \xrightarrow{j_\alpha^*} H\mathcal{S}(\alpha^*\mathcal{U}_G) \xrightarrow{\alpha_*} G\mathcal{S}\mathcal{U}_G$, where j_α^* is the contravariant change of universe.

Now, let X_G be a given $\mathcal{L}(\mathcal{U}_G)$-space. Then, by "neglect of structure" one obtains an $\mathcal{L}(\mathcal{U}_H)$-space $X_H = \alpha^*_{G,H} X_G$ for each $H \in \mathcal{G}$. Notice that the (non-equivariant) operad $\mathcal{L}(\mathcal{U}_H)^H$ of H-equivariant isometries acts on X_H^H and can be thought of as a suboperad of $\mathcal{L}(\mathcal{U}_e)$ via the non-equivariant identification of $\mathcal{U}_H$ with $\mathcal{U}_e$. It follows that the inclusion $X_H^H \hookrightarrow X_e$ is a map of $\mathcal{L}(\mathcal{U}_H)^H$-spaces.

Let us denote by $\mathbf{k}_H$ the spectrum associated to X_H via Theorem 2.2.3. The following result applies to various situations in equivariant homotopy theory, as we shall see later on.

Proposition 2.3.1. *Let X be an $\mathcal{L}(\mathcal{U}_G)$-space and suppose that for each $H \in \mathcal{G}$ one has a map $f_H : X_e \longrightarrow X_H^H$ of $\mathcal{L}(\mathcal{U}_H)^H$- spaces so that the composition $X_e \xrightarrow{f_H} X_H^H \xrightarrow{i_H} X_e$ is a homotopy equivalence in the category of $\mathcal{L}(\mathcal{U}_H)^H$-spaces. Then the family $\{\mathbf{k}_H \in H\mathcal{S}(\mathcal{U}_H) \mid H \in \mathcal{G}\}$ of associated spectra to $\{X_H \mid H \in \mathcal{G}\}$ satisfies:*

a. *For each map $\theta : H \longrightarrow H'$ in $\mathcal{G}$ there is a map of spectra $\eta_\theta : \theta^\sharp \mathbf{k}_{H'} \longrightarrow \mathbf{k}_H$ which is the identity if θ is the identity and an equivalence if θ is an inclusion.*

b. *The diagram*

$$\begin{array}{ccc} \beta^\sharp\alpha^\sharp\mathbf{k}_N & \xrightarrow{\beta^\sharp\eta_\alpha} & \beta^\sharp\mathbf{k}_H \\ {\scriptstyle\cong}\big\downarrow & & \big\downarrow{\scriptstyle\eta_\beta} \\ (\alpha\beta)^\sharp\mathbf{k}_N & \xrightarrow{\eta_{\alpha\beta}} & \mathbf{k}_K. \end{array}$$

homotopy commutes for every $\alpha : H \longrightarrow N$ and $\beta : K \longrightarrow H$.

Proof. See Appendix A. □

Remark 2.3.2. A collection of spectra satisfying the properties described in Proposition 2.3.1 is called a **coherent family of spectra indexed on** $\mathcal{G}$ or a $\mathcal{G}$-spectrum in [14, II.8].

Corollary 2.3.3. *Under the hypothesis of the proposition, all spectra $\mathbf{k}_H \in H\mathcal{S}(\mathcal{U}_H)$, with $H \in \mathcal{G}$, are split.*

Proof. This follows directly from the appropriate commutative diagram associated with the maps $H \xrightarrow{\beta} \{e\} \xrightarrow{\alpha} H$; cf. [14, Ex. 8.7(i)]. □

2.4. $G\mathcal{I}_*(\mathcal{U})$**-monoids.** A simple, geometric way of constructing $\mathcal{L}(\mathcal{U})$-spaces, for a fixed G-universe $\mathcal{U}$, is given by $G\mathcal{I}_*(\mathcal{U})$-spaces. Start with the category $\mathcal{I}_*$ of finite dimensional hermitian vector spaces and linear isometries between them. Denote by $G\mathcal{I}_*$ the subcategory of $\mathcal{I}_*$ consisting of G-modules and G-module morphisms and denote by $G\mathcal{I}_*(\mathcal{U})$ the full subcategory of $G\mathcal{I}_*$ consisting of finite dimensional G-modules isomorphic to submodules of $\mathcal{U}$.

Definition 2.4.1. A $G\mathcal{I}_*(\mathcal{U})$-space (T,ω) is a continuous covariant functor[2]

$$T : G\mathcal{I}_*(\mathcal{U}) \to G\text{-}\mathcal{T}op$$

from $G\mathcal{I}_*(\mathcal{U})$ to the category of non-degenerately based G- spaces together with a (coherently) commutative, associative, and continuous natural transformation

$$\omega : T \times T \to T \circ \oplus$$

satisfying:

(a) If $x \in TV$ and if $1 \in T\{0\}$ is the basepoint, then $\omega(x,1) = x \in T(V \oplus \{0\}) = TV$.

(b) If $V = V' \oplus V''$, then the map $TV' \to TV$ given by $x \mapsto \omega(x,1)$ is a homeomorphism onto a closed subset.

(c) Each sum map $\omega : T(V) \times T(W) \to T(V \oplus W)$ is a G-map.

(d) Each evaluation map $e : \mathcal{I}_*(V,W) \times T(V) \to T(W)$ is a G-map.

One can formulate the notion of morphisms of $G\mathcal{I}_*(\mathcal{U})$-spaces as a compatible collection of G-maps making the appropriate diagrams commute, and hence one obtains the category $G\mathcal{I}_*$-$\mathcal{T}op$ consisting of $G\mathcal{I}_*$-spaces and their maps.

The output of the formalism of $G\mathcal{I}_*$-spaces is the following simple, albeit very useful, result of J. P. May [18].

Proposition 2.4.2. *For any $G\mathcal{I}_*(\mathcal{U})$-space T, the colimit $T(\mathcal{U}) := \varinjlim_{V \subset \mathcal{U}} T(V)$, is an $\mathcal{L}(\mathcal{U})$-space. Furthermore, the assignment $T \mapsto T(\mathcal{U})$ is functorial, i.e., sends maps of $G\mathcal{I}_*(\mathcal{U})$-spaces to maps of $\mathcal{L}(\mathcal{U})$-spaces.*

Remark 2.4.3. $\mathcal{I}_*$-functors $T : \mathcal{I}_* \to \mathcal{F}$ may have values on any symmetric monoidal category $\mathcal{F}$ which admits colimits, in which case the colimit $T(\mathcal{U})$ becomes an $\mathcal{L}(\mathcal{U})$-object with values in $\mathcal{F}$.

It is unnecessary to stress the relevance of ring spectra in homotopy theory and how rich a theory becomes when one has an E_∞-operad parameterizing the ring structure of a spectrum. A precise formulation of the notion of E_∞-ring spectra can be found in [19] and, for our purposes, it suffices only to exhibit particular instances where such spectra occur. We need only the concrete manifestation of multiplicative infinite loop space theory, which is explained in the next few paragraphs.

Let G-$\mathcal{A}tom$ be the subcategory of G-$\mathcal{T}op$ whose objects are abelian topological monoids on which G acts by monoid morphisms, and whose morphisms are equivariant continuous monoid homomorphisms.

Definition 2.4.4. A $G\mathcal{I}_*(\mathcal{U})$-monoid is a $G\mathcal{I}_*(\mathcal{U})$-space (T,ω) with values in G-$\mathcal{A}tom$, where ω is distributive over the monoid operation. The collection of $G\mathcal{I}_*(\mathcal{U})$-monoids, together with those $G\mathcal{I}_*(\mathcal{U})$-space morphisms which are spacewise monoid morphisms, form a subcategory $G\mathcal{I}_*$-$\mathcal{A}tom$ of the category of all $G\mathcal{I}_*(\mathcal{U})$-spaces.

[2]Recall that a **topological category** is one where both the objects and the set of morphisms are spaces, and whose structural operations such as composition and evaluation are continuous. A **continuous functor** between topological categories is one which induces continuous maps between morphism spaces. See [21] for details.

In the same fashion as the nonequivariant case [19], the resulting G-monoid $T(\mathcal{U}) = \varinjlim_{V\subset\mathcal{U}} T(V)$ becomes an equivariant $\mathcal{L}(\mathcal{U})$-ring space. We summarize, for future convenience the main consequences of this fact, which follow from the application of the infinite loop space machines described above, in the following statements.

Proposition 2.4.5. *Let G be a finite group, and let T be a $G\mathcal{I}_*(\mathcal{U})$-monoid, where $\mathcal{U}$ is a complete G-universe. Then there is an $\mathcal{L}(\mathcal{U})$-ring spectrum $\mathbf{k}_G^T$, together with a morphism $\eta : T(\mathcal{U}) \to \mathbf{k}_G^T(0)$ of $\mathcal{L}(\mathcal{U})$-ring spaces which is a G-group completion of the additive structure of $T(\mathcal{U})$. In particular, if $T(\mathcal{U})$ is additively G-group-complete then η is a G-homotopy equivalence.*

Any commutative topological ring R yields a group-complete "constant" $G\mathcal{I}_*(\mathcal{U})$-monoid $\underline{R}$, on which G acts trivially. We then say that a $G\mathcal{I}_*(\mathcal{U})$- monoid T is augmented over R if there is a morphism $T \to \underline{R}$ of $G\mathcal{I}_*(\mathcal{U})$-monoids.

Proposition 2.4.6. *Let $\phi : T \to \underline{\mathbb{Z}}$ be a $G\mathcal{I}_*(\mathcal{U})$-monoid augmented over $\mathbb{Z}$, and assume that $T(\mathcal{U})$ is additively G-group complete. Then ϕ induces a map of $\mathcal{L}(\mathcal{U})$-ring spaces $\phi : T(\mathcal{U}) \to \mathbb{Z}$ and $T(\mathcal{U})_1{:=}\phi^{-1}(1)$ is a $\mathcal{L}(\mathcal{U})$-subspace of T. Therefore $T(\mathcal{U})_1$ admits a multiplicative G-group completion $\eta : T(\mathcal{U})_1 \to \mathbf{m}_{\mathbf{G}}^{\mathbf{T}}(\mathbf{0})$ into the zero-th space of a G-spectrum $\mathbf{m}_{\mathbf{G}}^{\mathbf{T}}$. In particular, if $T(\mathcal{U})_1$ is a connected space, then η is a nonequivariant homotopy equivalence.*

We have thus seen how the machinery of $G\mathcal{I}_*$-monoids automatically produces examples of highly structured equivariant ring spectra, and in the next chapter we will exhibit various examples of such objects.

3. Constructions

In this chapter we present the central constructions of the paper. We introduce certain concrete, geometric $\mathcal{I}_*(\mathcal{U})$-functors with values both in *G-Top* and in *G-$\mathcal{A}$tom*, the category of abelian topological G-monoids and equivariant monoid morphisms. Our constructions involve the use of algebraic cycles on complex projective spaces. Recall that an algebraic p-cycle σ on $\mathbb{P}^n$ is a finite formal sum $\sigma = \sum n_i V_i$, $n_i \in \mathbb{Z}$, where V_i is an irreducible p-dimensional subvariety of $\mathbb{P}^n$. An algebraic cycle is said to be *effective* if its coefficients are positive integers. The collection $\mathcal{C}_{p,d}(\mathbb{P}^n)$, of effective p-cycles on $\mathbb{P}^n$ of degree d, forms a projective variety called a Chow variety. The degree of a p-cycle $\sigma = \sum n_i V_i$ is defined as $\deg \sigma = \sum n_i \deg V_i$, where $\deg V_i \in \mathbb{Z}$ is the topological degree of V_i determined as the fundamental class $[V_i] \in H_{2p}(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}$, which coincides with the number of points in the intersection of V_i with a generic linear subspace of $\mathbb{P}^n$ of complementary dimension. We refer the reader to [11] and [17] for further details on topological properties of spaces of algebraic cycles.

3.1. **The Chow monoid functor.**

Let us fix, momentarily, a compact Lie group G and its universe $\mathcal{U} = \mathcal{U}_G$.

Definition 3.1.1. A. Given an hermitian inner product space V of dimension v, let

$$\mathcal{C}(V) := \coprod_{d\geq 0} \mathcal{C}^v_d(\mathbb{P}(V\oplus V)) \tag{3.1}$$

be the **Chow monoid** $\mathcal{C}^v(\mathbb{P}(V\oplus V))$ of effective algebraic cycles of codimension v in $\mathbb{P}(V\oplus V)$, with distinguished element $1 = \mathbb{P}(V\oplus\{0\})$. Here $\mathcal{C}^v_d(\mathbb{P}(V\oplus V))$ denotes the effective cycles of degree d and codimension v in $\mathbb{P}(V\oplus V)$. In dimension 0 we set $\mathcal{C}(0)=\mathbb{N}$ with distinguished element 1.

B. For a linear isometric embedding $f: V\to W$, we define $\mathcal{C}(f):\mathcal{C}(V)\to\mathcal{C}(W)$ on a codimension v cycle c by

$$\mathcal{C}(f)c \;=\; \mathbb{P}\left(W - f(V)\oplus\{0\}\right) \sharp_{\mathbb{C}} (f\oplus f)_*(c), \tag{3.2}$$

where $W-f(V)$ denotes the orthogonal complement of $f(V)$ in W and where $\sharp_{\mathbb{C}}$ denotes the complex join, which is defined on any pair of cycles which live in disjoint linear subspaces of $\mathbb{P}(V\oplus V)$. Recall that the complex join is given in homogeneous coordinates by taking the direct product of the homogeneous (conical) varieties. In particular, it is a strictly associative pairing. Note that $\mathcal{C}(f)(1)=1$.

C. Using the complex join we define a pairing $\omega:\mathcal{C}V\times\mathcal{C}V'\to\mathcal{C}(V\oplus V')$ by setting

$$\omega(c,c') \;=\; \tau_*(c\,\sharp_{\mathbb{C}}\,c') \tag{3.3}$$

where $\tau: V\oplus V\oplus V'\oplus V' \to V\oplus V'\oplus V\oplus V'$ is the shuffle map which interchanges the two middle factors. The pairing ω is *strictly associative, coherently commutative and continuous.*

The "diagonal" action of G in $V\oplus V$ induces an action of G on the Chow monoid $\mathcal{C}^v(\mathbb{P}(V\oplus V))$ which makes it into a G-$\mathcal{A}tom$. For an element $g\in G$ and $V\in\mathcal{I}_*(\mathcal{U})$ we let

$$g_\sharp:\mathcal{C}(V)\to\mathcal{C}(V) \tag{3.4}$$

denote the action of g on the cycles of $\mathbb{P}(V\oplus V)$ via the given representation. We make $\mathcal{C}^0(\mathbb{P}(0\oplus 0))$ into a trivial G-set.

Proposition 3.1.2. *For every compact Lie group G, the functor*

$$(\mathcal{C}_G,\omega): G\mathcal{I}_*(\mathcal{U}_G)\to G\text{-}\mathcal{A}tom$$
$$V\mapsto\mathcal{C}(V)$$

is a $G\mathcal{I}_*(\mathcal{U}_G)$-monoid which is augmented over $\mathbb{Z}$.

Proof. The fact that $\mathcal{C}$ is an $\mathcal{I}_*$-monoid in the nonequivariant situation is shown in [3], and the degree map is easily seen to give the desired augmentation. Therefore we just need to show the following:

Step 1: *The points $\underline{0}$ and $\underline{1}$ in $\mathcal{C}(V)$ are fixed by the action of G.*

Since $V\oplus\{0\}$ is a subrepresentation of $V\oplus V$ then $\mathbb{P}(V\oplus\{0\})$ is an invariant subspace of $\mathbb{P}(V\oplus V)$ under the action of G, and hence $1{:=}\mathbb{P}(V\oplus\{0\})$ is fixed as a cycle in $\mathcal{C}(V)$. The fact that $\underline{0}$ is fixed under the action follows from its definition.

Step 2: *The evaluation maps*

$$e : \mathcal{I}_*(V,W) \times \mathcal{C}(V) \to \mathcal{C}(W)$$

are G-maps.

Recall that G acts by isometries on V and $W \in \mathcal{U}$ and by conjugation on $\mathcal{I}_*(V,W)$. In other words, given $f \in \mathcal{I}_*(V,W)$ and $g \in G$, then $gf : V \to W$ is given by the formula $(gf)(v){:=}g(f(g^{-1}v))$ for $v \in V$. As a consequence one immediately gets, for all $g \in G$:

$$g\,(W - f(V)) \,=\, W - (gf)(V) \tag{3.5}$$

Now, for $f \in \mathcal{I}_*(V,W)\,, g \in G$ and $c \in \mathcal{C}(V)$ one has:

$$\begin{aligned} g_\sharp\,(e(f,c)) \;&=\; g_\sharp\,(\mathcal{C}(f)(c)) = \; g_\sharp\,(\mathbb{P}\,((W - f(V)) \oplus \{0\})\;\sharp_{\mathbb{C}}\,(f \oplus f)_*(c)) \\ &=\; g_\sharp \mathbb{P}\,((W - f(V)) \oplus \{0\})\;\sharp_{\mathbb{C}}\, g_\sharp\,((f \oplus f)_*(c)) \\ &=\; \mathbb{P}\,((W - (gf)(V)) \oplus \{0\})\;\sharp_{\mathbb{C}}\,((gf) \oplus (gf))_*\,(g_\sharp c) \\ &=\; e(gf\,, g_\sharp c), \end{aligned}$$

where the fourth equality follows from 3.5 and from the action by conjugation of G on $\mathcal{I}_*(V,W)$.

Step 3: *The Whitney sum*

$$\omega : \mathcal{C}(V) \times \mathcal{C}(W) \to \mathcal{C}(V \oplus W)$$

is a G-map, for all V , $W \in \mathcal{I}_*(\mathcal{U})$ and $V \perp W$.

Given $c \in \mathcal{C}(V)$, $c' \in \mathcal{C}(W)$ and $g \in G$ one has:

$$\begin{aligned} g_\sharp \omega(c,c') \;&=\; g_\sharp\,(\tau(c\,\sharp_{\mathbb{C}}\,c')) \\ &=\; \tau(g_\sharp c\,\sharp_{\mathbb{C}}\,g_\sharp c') \\ &= \omega(g_\sharp c, g_\sharp c') \end{aligned}$$

□

The construction of the $G\mathcal{I}_*(\mathcal{U}_G)$-monoid $\mathcal{C}_G$ presents an exceptional behavior with respect to subgroups. More precisely, if $\alpha : H \hookrightarrow G$ is the inclusion of a subgroup, then $\alpha^*\mathcal{C}_G(\mathcal{U}_G) = \mathcal{C}_H(\mathcal{U}_H)$ under the identification of $\mathcal{U}_H$ with $\alpha^*\mathcal{U}_G$. Furthermore, if $V \subset \mathcal{U}_H^H$ then $\mathcal{C}_H(V) = \mathcal{C}_H(V)^H$ and the isometry $j_\beta : \beta^*\mathcal{U}_e \to \mathcal{U}_H$, associated to the map $\beta : H \to \{e\}$ in (2.1), induces a natural map $\tau_H : \mathcal{C}_e(\mathcal{U}_e) \to \mathcal{C}_H(\mathcal{U}_H)^H$ of $\mathbb{Z}$-augmented $\mathcal{L}(U_H)^H$-monoids.

Proposition 3.1.3. *The composition $\mathcal{C}_e(\mathcal{U}_e) \xrightarrow{\tau_H} \mathcal{C}_H(\mathcal{U}_H)^H \to \mathcal{C}_H(\mathcal{U}_H)$ is a non-equivariant homotopy equivalence for each subgroup H of G.*

Proof. First recall that one can assume that the H-linear isometric embedding $j_\beta : \beta^*\mathcal{U}_e \to \mathcal{U}_H$ is induced by a (non-equivariant) identification of $\mathcal{U}_e$ with $\mathcal{U}_H^H$, followed by an inclusion. We use the letter $\mathcal{C}$ to denote the Chow monoid functor when no mention to group actions is necessary, and identify $\mathcal{C}_e(\mathcal{U}_e)$ with $\mathcal{C}(\mathcal{U}_H^H)$. Therefore, one just needs to prove that the map $\mathcal{C}(\mathcal{U}_H^H) \to \mathcal{C}(\mathcal{U}_H)$ induced by the inclusion $\mathcal{U}_H^H \hookrightarrow \mathcal{U}_H$ is a non-equivariant homotopy equivalence.

Let us denote $\mathcal{U}_H$ by $\mathcal{U}$ and $\mathcal{U}_H^H$ by $\mathcal{U}_e$, and write $\mathcal{U} = \mathcal{U}_e^{\perp} \oplus \mathcal{U}_e$. Fix an isometric isomorphism $l : \mathcal{U}_e \to \mathcal{U}_e \oplus \mathcal{U}_e$ and observe that $L{:=}1_{\mathcal{U}_e^{\perp}} \oplus l : \mathcal{U} \to \mathcal{U} \oplus \mathcal{U}_e$ is an H-linear isometric isomorphism. Let $i_0, i_1 : \mathcal{U}_e \hookrightarrow \mathcal{U}_e \oplus \mathcal{U}_e$ denote the inclusions in the first and second factors, respectively, and let $\gamma_0, \gamma_1 : [0,1] \to \mathcal{I}(\mathcal{U}_e, \mathcal{U}_e \oplus \mathcal{U}_e)$ be paths in the contractible space of linear isometries $\mathcal{I}(\mathcal{U}_e, \mathcal{U}_e \oplus \mathcal{U}_e)$ joining i_0 and i_1 to l, respectively.

Notice that $1_{\mathcal{U}_e^{\perp}} \oplus \gamma_0$ induces an equivariant homotopy from the inclusion $j_0 : \mathcal{U} \hookrightarrow \mathcal{U} \oplus \mathcal{U}_e$ to the H-linear isomorphism L, introduced above. Furthermore, under the identification of $\mathcal{U}$ with $\mathcal{U} \oplus \mathcal{U}_e$ via L, one sees that $1_{\mathcal{U}_e^{\perp}} \oplus \gamma_1$ induces a homotopy between the inclusion $\mathcal{U}_e \hookrightarrow \mathcal{U}$ and the inclusion in the second factor $j_1 : \mathcal{U}_e \hookrightarrow \mathcal{U} \oplus \mathcal{U}_e$.

Given a linear isometry $f \in \mathcal{I}(\mathcal{U}, \mathcal{U} \oplus \mathcal{U}_e)$, it naturally defines a monoid morphism $f_* : \mathcal{C}(\mathcal{U}) \to \mathcal{C}(\mathcal{U} \oplus \mathcal{U}_e)$, as follows. For each $V \subset \mathcal{U}$, f induces a natural map $f_{*,V} : \mathcal{C}(V) \to \mathcal{C}(f(V))$ which is seen to be a map of directed systems, once one observes that for $V \subset W$, one has $f(W - V) = f(W) - f(V)$, since f is an isometry. It is not hard to see that the assignment $f \mapsto f_*$ induces a continuous map $\Gamma : \mathcal{I}(\mathcal{U}, \mathcal{U} \oplus \mathcal{U}_e) \times \mathcal{C}(\mathcal{U}) \to \mathcal{C}(\mathcal{U} \oplus \mathcal{U}_e)$.

The composition

$$[0,1] \times \mathcal{C}(\mathcal{U}) \xrightarrow{1_{\mathcal{U}_e^{\perp}} \oplus \gamma_1} \mathcal{I}(\mathcal{U}, \mathcal{U} \oplus \mathcal{U}_e) \times \mathcal{C}(\mathcal{U}) \xrightarrow{\Gamma} \mathcal{C}(\mathcal{U} \oplus \mathcal{U}_e) \xrightarrow{l_*^{-1}} \mathcal{C}(\mathcal{U})$$

gives an equivariant homotopy between $\mathcal{C}(\mathcal{U}) \hookrightarrow \mathcal{C}(\mathcal{U} \oplus \mathcal{U}_e)$ and $\mathcal{C}(\mathcal{U}) \xrightarrow{l_*} \mathcal{C}(\mathcal{U} \oplus \mathcal{U}_e)$.

At this point we can identify, non-equivariantly, $\mathcal{U}$ with $\mathcal{U}_e$ and $\mathcal{U}_e$ with $\mathcal{U}_e^{\perp}$ in the formula above, to avoid repetition of arguments, and reinterpret the same homotopy as providing the desired homotopy equivalence. □

Remark 3.1.4. Notice that $\mathcal{C}_H(\mathcal{U}_H)$ is non-equivariantly identified with $\mathcal{C}_e(\mathcal{U}_e)$ for every $H \in \mathcal{G}$, and any group G. In particular, the Proposition above shows that $\mathcal{C}_G(\mathcal{U}_G)$ is an $\mathcal{L}(\mathcal{U}_G)$-space satisfying the conditions of Proposition 2.3.1.

3.2. **The Grassmann functor and equivariant K-theory.** We start by observing that the Chow variety $\mathcal{C}_1^v(\mathbb{P}(V \oplus V))$ of cycles of degree 1 in $\mathbb{P}(V \oplus V)$ is precisely the Grassmannian variety $\mathrm{Gr}^v(V \oplus V)$, of v-planes in $V \oplus V$, where $v = \dim_{\mathbb{C}} V$.

In the setting of the previous constructions, the complex join, when restricted to linear subspaces, becomes the direct sum of planes. As a consequence, the following result is just an easy corollary of the previous Proposition.

Proposition 3.2.1. *For each compact Lie group G, the assignment*

$$(Gr, \oplus) : G\mathcal{I}_*(\mathcal{U}_G) \to G\text{-}\mathcal{T}op$$

given by

$$V \mapsto \mathcal{C}_1^v(\mathbb{P}(V \oplus V)) \cong Gr^v(V \oplus V),$$

defines a $G\mathcal{I}_*(\mathcal{U}_G)$-subspace of $\mathcal{C}_G$. Furthermore, the map τ_G of Proposition 3.1.3 restricts to a map $\tau_G : Gr(\mathcal{U}_e) \to Gr(\mathcal{U}_G)^G$ whose composition with the inclusion $Gr(\mathcal{U}_G)^G \to Gr(\mathcal{U}_G)$ is also a non-equivariant homotopy equivalence.

Proof. The first assertion follows the nonequivariant analogue, cf. [3] once we observe that $Gr(V)$ is a G-subspace of $\mathcal{C}(V)$, for every G-module V and that the complex join operation is equivariant. Furthermore, the homotopy equivalence obtained in Proposition 3.1.3 restricts to a homotopy equivalence in the level of the Grassmann subspaces, since it preserves degrees. □

Remark 3.2.2. Notice that Gr is not a $G\mathcal{I}_*$-sub**monoid** of $\mathcal{C}$, but only a subspace. However, the assignment of the infinite symmetric product $SP(Gr(V))$ to V, is naturally identified with a $G\mathcal{I}_*$-submonoid of $\mathcal{C}$. It is important to note that $SP(Gr(V)^G)$ is not necessarily equal to $SP(Gr(V))^G$. Actually, after group completion this difference is explained in terms of Bredon homology in [15].

Remark 3.2.3. It follows immediately from 3.2.1 and Theorem 2.2.3 that the direct limit

$$B\mathcal{U}_G := Gr(\mathcal{U}_G) = \varinjlim_V Gr(V)$$

is an $\mathcal{L}(\mathcal{U}_G)$-space and therefore inherits the sturcture of an equivariant infinite loop space. In particular, there is a G- spectrum $\mathfrak{ku}_G(*)$ and a G-map

$$B\mathcal{U}_G \longrightarrow \mathfrak{ku}_G(0)$$

into the zero'th space of the spectrum, which is a G-group completion.

The spectrum $\mathfrak{ku}_G$ corresponds to connective equivariant K-theory (cf. [14]). For completeness we show here directly that the zero'th space $B\mathcal{U}_G$ classifies the functor $K_G(X)$. We begin by recalling the basic definitions and results (cf. [22]). Let X be a compact G-space, and denote by $\mathcal{V}_G(X)$ the set of isomorphism classes of complex G-vector bundles over X. Under the Whitney sum $\oplus$ this is an abelian monoid whose naïve group completion $K_G(X)$ is called the **equivariant K-theory of X**. Given a G-module V, we denote by $\mathbb{V}$ the "trivial" G- vector bundle $X \times V \to X$. Elements in $K_G(X)$ can be canonically identified with stable equivalence classes in $\mathcal{V}_G(X)$, where E, E' are called **stably equivalent** if there exist G-modules V, V' so that $E \oplus \mathbb{V} \cong E' \oplus \mathbb{V}'$. This is a consequence of the following.

Proposition 3.2.4. *([22], Prop. 2.4) If E is a G-vector bundle over a compact space X, there is a G-module V and a G-vector bundle $E^{\perp}$ such that $E \oplus E^{\perp} \cong \mathbb{V}$.*

There is an equivariant augmentation $\phi : K_G(X) \to \mathbb{Z}$ given by the virtual rank, and we set $K_G(X)_o = \ker(\phi)$.

Given a G-module W, let ξ_W denote the "universal" G- vector bundle over $Gr(W)$ whose fibre at V is $V^{\perp}$. Given a $G\mathcal{I}_*$- map $j : U \to W$, one has

$$j^*\xi_W \;\cong\; \xi_U \oplus (\mathbb{W} - j(\mathbb{U})),$$

where $\mathbb{W} - f(\mathbb{U})$ is the "trivial" bundle associated to the orthogonal complement of $\{0\} \times j(U)$ in $\{0\} \times W$.

Consider now a G-map $f : X \to B\mathcal{U}_G$. Since X is compact and $B\mathcal{U}_G$ carries the compactly generated topology, there is a G-module $W \subset \mathcal{U}_G$ and a G-map $f_W : X \to Gr(W)$ which represents f in the limit. We associate to f the element

$$f_W^* - \mathbb{W} \in K_G(X). \tag{3.6}$$

This element *depends only on the G-homotopy class* of f ([22], Prop. 1.3). Furthermore, this element restricts to zero at the base point. Thus, the assignment (3.6) determines a well-defined homomorphism

$$i : [X^+, B\mathcal{U}_G]_G \ \to \ K_G(X), \tag{3.7}$$

where X_+ denotes the union of X with a disjoint base point.

Proposition 3.2.5. *The following sequence is exact:*

$$0 \to [X_+, B\mathcal{U}_G]_G \xrightarrow{i} K_G(X) \xrightarrow{\phi} \mathbb{Z} \to 0,$$

and hence $[X_+, B\mathcal{U}_G]_G \cong K_G(X)_o$.

Proof. It is clear from its construction that image$(i) \subseteq \ker(\phi)$. To show surjectivity onto $\ker(\phi)$, consider $E - E' \in \ker(\phi)$. Then $\mathrm{rank}(E) = \mathrm{rank}(E') = m$ and we extend E and E' to X_+ by defining them to be the trivial representation of rank m at the disjoint base point. By Proposition 3.2.4 there is a G-vector bundle $E'^{\perp}$ over X_+ and a G-module U so that $E' \oplus E'^{\perp} \cong \mathbb{U}$. By adding and subtracting this element in $E - E'$ we may assume that $E' = \mathbb{U}$.

Similarly we may choose $E^{\perp}$ over X_+ with $E \oplus E^{\perp} \cong \mathbb{W}$ for some G-module W. The inclusion $E^{\perp} \subset \mathbb{W}$ induces a G-map $f : X \to \mathrm{Gr}_n(W)$ $(n = \mathrm{rank}(E^{\perp}))$ which classifies E in the sense that $f^*\xi_W = E$, where ξ_W is the universal bundle as above. Let $\phi : \mathrm{Gr}_n(W) \to Gr(W \oplus U)$ be defined by $\phi(L) = L \oplus \{0\} \oplus \{0\} \oplus U \ \subset \ (W \oplus U) \oplus (W \oplus U)$. Then one has $(\phi \circ f)^*\xi_{W\oplus U} \cong E \oplus \mathbb{W}$, and hence $i([\phi \circ f]) = (E \oplus \mathbb{W}) - (\mathbb{W} \oplus \mathbb{U}) = E - \mathbb{U}$ (restricted to X).

Proof of the injectivity of i is analogous to the non-equivariant case. ☐

Remark 3.2.6. One knows that $R(G)$ is generated as an abelian group by the classes $\xi_1, \dots, \xi_m$ of the irreducible representations of G. Later on, especially in Section 5, the group ring of this abelian group will play an important role in our computations. For this reason, we shall write $R(G)$ *multiplicatively*, i.e., its elements will be expressed as Laurent monomials $\xi_1^{n_1}\xi_2^{n_2}\cdots\xi_m^{n_m}$ for $n_i \in \mathbb{Z}$.

Example 3.2.7. Let $X = pt$ so that $X^+ = S^0$ and $K_G(X) = R(G)$ and $K_G(X)_o = R(G)_o =$ the kernel of the augmentation map $R(G) \to \mathbb{Z}$. Then the previous proposition shows that

$$\pi_0((B\mathcal{U}_G)^G) = [S^0, (\mathrm{B}\mathcal{U}_G)^G] = [S^0, B\mathcal{U}_G]_G \cong R(G)_o. \tag{3.8}$$

Note that in the multiplicative notation above we can write $R(G)_o$ as

$$R(G)_o \ = \ \{\xi_1^{n_1}\cdots\xi_m^{n_m} \mid \sum n_i \dim \xi_i = 0\} \tag{3.9}$$

i.e., as the multiplicative group of "Laurent monomials of weighted degree zero" in the irreducible representations of G.

3.3. **Topological groups of cycles.** For any vector space V the monoid $\mathcal{C}(V)$ completes, in the naïve Grothendieck sense, to become the free abelian group $\mathcal{Z}(V)$ of all codimension-v cycles on $\mathbb{P}(V \oplus V)$ with topology induced from $\mathcal{C}(V)$. In [16] it is proved that this **naïve topological group completion** $\mathcal{C}(V) \to \mathcal{Z}(V)$ is equivalent to the homotopy-theoretic one $\mathcal{C}(V) \to \Omega B\mathcal{C}(V)$.

In this section we carry this result over to the current context, proving that the induced map of $\mathcal{L}(\mathcal{U}_G)$-spaces $\mathcal{C}(\mathcal{U}_G) \to \mathcal{Z}(\mathcal{U}_G)$ is a *G-group completion* equivalent to the one constructed above. This result is useful for computations and localization.

To begin we note that the assignment $V \mapsto \mathcal{Z}(V)$, together with the join pairing as above, defines a $G\mathcal{I}_*(\mathcal{U}_G)$-monoid which is additively group complete. Furthermore, from [16] we have the following.

Proposition 3.3.1. *The induced map $\mathcal{C}(\mathcal{U}) \longrightarrow \mathcal{Z}(\mathcal{U})$ of direct limit spaces is a non-equivariant additive group completion.*

Proof. Homotopy and homology group functors commute with direct limits. □

Let us consider the equivariant case. The following result follows from the freeness of $\mathcal{C}(V)$:

Lemma 3.3.2. *For every G-module V and subgroup $H \leq G$, the naïve topological group completion of the monoid $\mathcal{C}(V)^H$ of effective cycles fixed by H is the group of fixed cycles $\mathcal{Z}(V)^H$ of the naïve group completion $\mathcal{Z}(V)$ of $\mathcal{C}(V)$. In particular, the naïve group completion of $\mathcal{C}(V)^H$ is a Hausdorff topological group with the natural quotient topology.*

Proof. Obvious. □

For each $h \in H$, the action $h : \mathcal{C}(V) \to \mathcal{C}(V)$ preserves degree and is an algebraic map. Therefore, the intersection $\mathcal{C}^v_d(\mathbb{P}(V \oplus V)) \cap \mathcal{C}(V)^H$ is an algebraic variety, and the restriction of the monoid operation to the pieces of bounded degree is an algebraic map. This implies that $\mathcal{C}(V)^H$ is a *properly c-graded monoid* in the sense of Definition 4.1 of [16]. Algebraically $\mathcal{C}(V)^H$ is a *free* monoid generated by H-orbits of irreducible subvarieties of $\mathcal{C}(V)$. Therefore, by [16, Thm. 4.4] the naïve group completion $\mathcal{C}(V)^H \to \mathcal{Z}(V)^H$ is equivalent to the homotopy-theoretic one. Thus the assignment $V \to \mathcal{Z}(V)$ defines a $\mathbb{Z}$-augmented $G\mathcal{I}_*(\mathcal{U}_G)$-space taking values in the category of abelian topological groups, and the natural inclusions $\mathcal{C}(V) \hookrightarrow \mathcal{Z}(V)$ induce a map of augmented $G\mathcal{I}_*(\mathcal{U}_G)$-monoids so that for every $H \in \mathcal{G}$ and $V \subset \mathcal{U}_H$, the map $\mathcal{C}(V)^H \hookrightarrow \mathcal{Z}(V)^H$ is an additive group completion. This gives the following.

Proposition 3.3.3. *The map of $\mathcal{L}(\mathcal{U}_G)$-spaces $\mathcal{C}(\mathcal{U}_G) \to \mathcal{Z}(\mathcal{U}_G)$ is a G-group completion.*

Remark 3.3.4.

a. The $G\mathcal{I}_*$-functors $(\mathcal{C}, \sharp)$ and $(\mathcal{Z}, \sharp)$ determine G-spectra $\mathbf{k}^{\mathcal{C}}_G(*)$ and $\mathbf{k}^{\mathcal{Z}}_G(*)$ respectively. Furthermore, the natural transformation $(\mathcal{C}, \sharp) \longrightarrow (\mathcal{Z}, \sharp)$ induces a morphism $j : \mathbf{k}^{\mathcal{C}}_G(*) \longrightarrow$

$\mathbf{k}_G^{\mathcal{Z}}(*)$ of G-spectra and a commutative diagram

$$\begin{array}{ccc} \mathcal{C}(\mathcal{U}) & \xrightarrow{\alpha_0} & \mathbf{k}_G^{\mathcal{C}}(0) \\ {\scriptstyle j_0}\downarrow & & \downarrow{\scriptstyle j} \\ \mathcal{Z}(\mathcal{U}) & \xrightarrow{\alpha} & \mathbf{k}_G^{\mathcal{Z}}(0) \end{array}$$

where by Proposition 2.4.5, α_0 is an additive G-group completion and α is a G-homotopy equivalence. By 3.3.3 j_0 is a G-group completion. It follows that $j : \mathbf{k}_G^{\mathcal{C}}(*) \longrightarrow \mathbf{k}_G^{\mathcal{Z}}(*)$ is an equivalence of G-spectra.

b. The functors $\mathcal{Z}_G$ suffice for the construction of all spectra we shall deal with. Nevertheless, we need the properties of the Chow monoid functor as an algebraic geometric object, together with the Proposition 3.3.3, in order to perform the computations in Chapter 5.

c. It is clear from its very definition that the homotopies introduced in the proof of Proposition 3.1.3 extend to **degree preserving** homotopies in the level of $\mathcal{Z}_G$. In particular, Proposition 3.1.3 remains true if one replaces $\mathcal{C}_G$ by $\mathcal{Z}_G$ in its statement.

Our next step is the analysis of the inclusion $\mathcal{C}(V)^G \subset \mathcal{C}(V)^H$ for a subgroup $H \leq G$. In this case $\mathcal{C}(V)^G$ is a closed submonoid of $\mathcal{C}(V)^H$, and $\mathcal{C}(V)^G \cap \mathcal{C}_d^v(\mathbb{P}(V \oplus V))$ is a subvariety of $\mathcal{C}(V)^H \cap \mathcal{C}_d^v(\mathbb{P}(V \oplus V))$.

We prove the following

Proposition 3.3.5. *The sequence*

$$\mathcal{Z}(V)^G \to \mathcal{Z}(V)^H \to \mathcal{Z}(V)^H/\mathcal{Z}(V)^G$$

is a principal fibration. In particular, after taking limits, one gets a principal fibration

$$\mathcal{Z}(\mathcal{U})^G \to \mathcal{Z}(\mathcal{U})^H \to \mathcal{Z}(\mathcal{U})^H/\mathcal{Z}(\mathcal{U})^G$$

Proof. The proposition would be a straightforward consequence of Theorem 5.2 of [16] if the following condition were satisfied: $\mathcal{C}(V)^G$ *is freely generated by a subset of the generators of* $\mathcal{C}(V)^H$. However, this assertion is not true.

On the other hand, the role played by such condition in the proof of the aforementioned theorem was to show that the group completion $\mathcal{Z}(V)^G$ is a closed subgroup of $\mathcal{Z}(V)^H$ and that the quotient $\mathcal{Z}(V)^H/\mathcal{Z}(V)^G$ is filtered by cofibrations under the quotient topology. This is clearly satisfied in the present situation.

Now, the proof of the existence of a local cross section for the quotient map $\mathcal{Z}(V)^H \to \mathcal{Z}(V)^H/\mathcal{Z}(V)^G$ follows *mutatis mutandis* the proof of Theorem 5.2 of [16]. □

3.4. **Further constructions and KO-theory.** Let T be a $G\mathcal{I}_*(\mathcal{U}_G)$-monoid and let K be a finite group which acts on T in the sense that it acts on $T(V)$ by monoid morphisms, for all $V \in \mathcal{I}_*(\mathcal{U}_G)$, and its action commutes with the action of G and is compatible with the structure maps of T.

Example 3.4.1. Let K be the center of G.

One can naturally define two $\mathcal{I}_*(\mathcal{U}_G)$-submonoids of T associated to the action of K as follows.

Definition 3.4.2.

$\boldsymbol{T^K}$: (**Fixed point functor**) For every $V \in \mathcal{I}_*(\mathcal{U}_G)$ let $T^K(V)$ denote the submonoid of $T(V)$ consisting of those elements in $T(V)$ which are fixed by the group K.

$\boldsymbol{KT}$: (**Averaged elements functor)** For every $V \in \mathcal{I}_*(\mathcal{U}_G)$ let $KT(V)$ denote the submonoid of T^K consisting of the "averaged" elements over K. More precisely,

$$KT(V) := \{ \sum_{z \in K} z_\sharp \sigma \mid \sigma \in T(V) \}.$$

Proposition 3.4.3.

a: *T^K is a $G\mathcal{I}_*(\mathcal{U}_G)$-submonoid of T.*

b: *KT is a $G\mathcal{I}_*(\mathcal{U}_G)$-submonoid of T^K. Furthermore, KT is an ideal of T^K, in the sense that*

$$\omega(T^K(V)\,,\, KT(W)) \subseteq KT(V \oplus W),$$

for all $V, W \in \mathcal{I}_*(\mathcal{U}_G)$.

Proof. **a:** Given $g \in G$ and $c\,,\, c' \in T(V)$ one has

$$g_\sharp(c + c') \;=\; g_\sharp c \,+\, g_\sharp c' \tag{3.10}$$

and

$$\begin{aligned} g_\sharp \circ \omega(c, c') \;&=\; g_\sharp\left(\tau(c \,\sharp_{\mathbb{C}}\, c')\right) \\ &=\; \tau \circ g_\sharp\left(c \,\sharp_{\mathbb{C}}\, c'\right) \;=\; \tau\left((g_\sharp c) \,\sharp_{\mathbb{C}}\, (g_\sharp c')\right) \\ &=\; \omega(g_\sharp c\,,\, g_\sharp c'). \end{aligned} \tag{3.11}$$

The combination of (3.10) and (3.11) proves the assertion.

b: For $c \in T^K(V)$ and $c' \in KT(W)$ one has, by definition: $h_\sharp c = c$ for all $h \in K$ and $c' = \sum_{h \in K} h_\sharp c''$, for some $c'' \in T(W)$.

Therefore, using 3.10 and 3.11 one obtains

$$\begin{aligned} \omega(c, c') \;&=\; \omega(c\,,\, \sum_{h \in K} h_\sharp c'') \\ &=\; \sum_{h \in K} \omega(c\,,\, h_\sharp c'') \;=\; \sum_{h \in K} \omega(h_\sharp c\,,\, h_\sharp c'') \\ &=\; \sum_{h \in K} h_\sharp \omega(c, c'') \in KT(V \oplus W). \end{aligned}$$

$\square$

Using the results above and the topological bar construction of [19, Defn. 2.2], one can define the following $\mathcal{I}_*(\mathcal{U}_G)$-monoid.

Definition 3.4.4. Given a group K acting on the $G\mathcal{I}_*(\mathcal{U}_G)$-monoid T as above, and $V \in \mathcal{I}_*(\mathcal{U}_G)$, let $(T/K)(V)$ be the monoid defined as

$$(T/K)(V) := B(T^K(V), KT(V), *),$$

where $B(-,-,-)$ denotes the toplogical triple bar construction, and where $KT(V)$ is acting on $T^K(V)$ by right translations.

Proposition 3.4.5. *The assignment $V \mapsto (T/K)(V)$ naturally inherits the structure of a $\mathcal{I}_*(\mathcal{U}_G)$-monoid from T.*

Proof. It follows straightforwardly from Proposition 3.4.3 and [19, §2]. □

A simple example of this set-up arises, in the same spirit of [3], when one uses the **real** linear isometries operad and Chow monoids to construct $RO(G)$-graded theories.

We denote by $\mathcal{I}_*(\mathcal{U}_G)$ the category of real finite-dimensional inner product spaces of a fixed real G-universe $\mathcal{U}$ and their linear isometries. Complexification (plus sesquilinear extension of inner products) identifies $\mathcal{I}_*(\mathcal{U}_G)$ as a subcategory of $\mathcal{I}_*(\mathcal{U}_G \otimes \mathbb{C})$. In particular, every $G\mathcal{I}_*(\mathcal{U}_G \otimes \mathbb{C})$-functor becomes a $G\mathcal{I}_*(\mathcal{U}_G)$-functor.

Consider now the Chow $G\mathcal{I}_*$-monoid $\mathcal{C}$ described in (3.1). Note that for any real orthogonal representation $V \subset \mathcal{U}_G$ the G-monoid $\mathcal{C}^v_G(\mathbb{P}(V_{\mathbb{C}} \oplus V_{\mathbb{C}}))$ (where $V_{\mathbb{C}} = V \otimes \mathbb{C}$) admits a $\mathbb{Z}/2\mathbb{Z}$ action induced by complex conjugation which commutes with the action of G. We now invoke Definition 3.4.4 with $K = \mathbb{Z}/2\mathbb{Z}$.

Definition 3.4.6. By the **real Chow functor** we mean $\mathcal{C}_{\mathbb{R}} := \mathcal{C}/(\mathbb{Z}/2\mathbb{Z})$ with its natural structure of a $\mathcal{I}_*(\mathcal{U}_G)$- monoid provided by 3.4.5.

In parallel fashion one can construct a **real Grassmann functor** $Gr_{\mathbb{R}}$ together with a natural transformation $(\mathrm{Gr}_{\mathbb{R}}, \oplus) \longrightarrow \mathcal{C}_{\mathbb{R}}, \sharp)$ of $G\mathcal{I}_*(\mathcal{U}_G)$- functors. This induces a map

$$Gr_{\mathbb{R}}(\mathcal{U}_G) \;\longrightarrow\; \mathcal{C}_{\mathbb{R}}(\mathcal{U}_G) \tag{3.12}$$

of $\mathcal{L}(\mathcal{U}_G)$-spaces. One checks directly that $Gr^v_{\mathbb{R}}(V)$ is exactly the Grassmannian of real codimension-v planes in $V \oplus V$ where $v = \dim_{\mathbb{R}}(V)$. Therefore, just as in §3.2 we may identify

$$BO(\mathcal{U}_G) := Gr_{\mathbb{R}}(\mathcal{U}_G) = \varinjlim{}_V Gr_{\mathbb{R}}(V)$$

with the classifying space for reduced KO_G-theory. In fact we have the following analogue of 3.2.5.

Proposition 3.4.7. *For any compact G-space X, the following sequence is exact:*

$$0 \to [X_+, BO(\mathcal{U}_G)]_G \xrightarrow{i} KO_G(X) \xrightarrow{\phi} \mathbb{Z} \to 0,$$

and hence $[X_+, BO(\mathcal{U}_G)]_G \cong KO_G(X)_o$ = the reduced KO_G-group of X.

4. Main Results

In this chapter we apply the machinery and constructions developed above in order to obtain the cohomology theories of primary interest in this paper, and to study their properties.

4.1. **The equivariant cohomology theories.**

Definition 4.1.1. For each finite group G, let $\mathfrak{Z}_G$ be the $\mathcal{L}(\mathcal{U}_G)$-ring-spectrum associated to the $G\mathcal{I}_*(\mathcal{U}_G)$- functor $\mathcal{Z}$ introduced in §3.3, and let $\mathfrak{M}_G$ denote the multiplicative spectrum associated to $\mathcal{Z}(\mathcal{U}_G)_1$, as in Proposition 2.4.6. Denote by $\delta_G : \mathfrak{Z}_G \to \mathbf{b}\mathfrak{Z}_G$ and $\epsilon_G : \mathfrak{M}_G \to \mathbf{b}\mathfrak{M}_G$ the natural maps into their associated Borel theories, respectively; cf. Example 2.1.3.

Remark 4.1.2. It follows from Proposition 3.3.3 and universal properties of group completions that $\mathfrak{Z}_G$ is the same spectrum as the one associated with the Chow monoid functor of Proposition 3.1.2.

We first describe the non-equivariant properties of our spectra.

Theorem 4.1.3.

a. *The underlying non-equivariant spectra* $\mathfrak{Z}$ *and* $\mathfrak{M}$ *associated to* $\mathfrak{Z}_G$ *and* $\mathfrak{M}_G$*, respectively, are independent of the group* G*.*

b. *The zero-th space* $\mathfrak{Z}(0)$ *of* $\mathfrak{Z}$ *is a weak product* $\prod_{j\geq 0} K(\mathbb{Z},2j)$ *of Eilenberg-MacLane spaces whose additive structure is the usual one, and whose multiplicative structure is induced by the cup product.*

c. *The map* $\mathcal{Z}(\mathcal{U})_1 \longrightarrow \mathfrak{M}(0)$ *is a homotopy equivalence and the underlying non-equivariant spectrum* $\mathfrak{M}$ *is the BLLMM spectrum of* [3]*. In particular,* $\mathfrak{M}(0) \cong 1 \times \prod_{j\geq 1} K(\mathbb{Z},2j)$ *and its infinite loop space structure enhances the cup product operation.*

Proof. **a.** Let $\alpha : \{e\} \hookrightarrow G$ be the inclusion. By definition, $\mathfrak{Z} = \alpha^\sharp \mathfrak{Z}_G$. On the other hand, it follows from Propositions 3.1.3 and 2.3.1, and Remark 3.1.4 that $\alpha^\sharp \mathfrak{Z}_G = \mathfrak{Z}_e$ is the spectrum associated to the non-equivariant Chow monoid functor $\mathcal{C}$ of Definition 3.1.1, independently of the group G. Similar argument applies to $\mathfrak{M}$.

b. This follows from the previous item, together with [11] and [13].

c. This follows from item (a) and [3]. □

The coherence properties with respect of changes of groups are expressed in the following result:

Theorem 4.1.4.

a. *For every finite group* G*, the natural map* $\mathcal{Z}_G(\mathcal{U}_G) \longrightarrow \mathfrak{Z}_G(0)$ *is an equivariant homotopy equivalence.*

b. *For a given finite group* G*, the collections of spectra* $\{\mathfrak{Z}_H \mid H \in \mathcal{G}\}$ *and* $\{\mathfrak{M}_H \mid H \in \mathcal{G}\}$ *form coherent families of spectra, in the sense of Remark 2.3.2. In particular, if* $H \in \mathcal{G}$ *and* $\alpha : H \longrightarrow G$ *is the inclusion, then* $\alpha^\sharp \mathfrak{Z}_G = \mathfrak{Z}_H$ *and* $\alpha^\sharp \mathfrak{M}_G = \mathfrak{M}_H$*, and both* $\mathfrak{Z}_G$ *and* $\mathfrak{M}_G$ *are split spectra.*

Proof. **a.** This follows from Propositions 2.4.5 and 3.3.3.

b. This follows from Propositions 3.1.3, 2.3.1 and Remark 3.3.4(c). □

The cohomology theories $\mathfrak{Z}_G^*$ and $\mathfrak{M}_G^*$ (see Definition 2.1.5) are quite rich and intriguing, and even the computation of their coefficients present several technical difficulties. On the other hand, their associated Borel theories are much more approachable and amenable to calculations, as the following results indicate.

Theorem 4.1.5. *Let* X *be a compact* G*-space.*

a. *The zero-th cohomology ring* $\mathbf{b}\mathfrak{Z}_G^0(X)$ *is isomorphic to the completion at the augmentation ideal of the ordinary even Borel cohomology ring of* X*, with coefficients in* $\mathbb{Z}$*. In* *other words,*

$$\mathbf{b}\mathfrak{Z}^0_G(X) \cong \prod_{j\geq 0} H^{2j}_G(X,\mathbb{Z}) \tag{4.1}$$

with the multiplicative structure induced by the cup product.

b. *If $k>0$ is a positive integer, then*

$$\mathbf{b}\mathfrak{Z}^{-k}_G(X) \cong \prod_{j\geq 0} H^{2j-k}_G(X,\mathbb{Z}) \tag{4.2}$$

as a module over $\mathbf{b}\mathfrak{Z}^0_G(X)$.

Proof. **a.** The usual adjunction gives $[X_+, \mathbf{b}\mathfrak{Z}_G(0)]_G \cong [X_+ \times EG_+, \mathfrak{Z}_G(0)]_G$ and the fact that the spectrum is split shows that the latter group is $[X_+ \wedge_G EG_+, \mathfrak{Z}(0)]$. It follows from Theorem 4.1.3(b) that $\mathbf{b}\mathfrak{Z}^0_G(X) \cong \prod H^{2j}(X\times_G EG,\mathbb{Z})$, proving the assertion.

b. For $k>0$, the same argument shows that $\mathbf{b}\mathfrak{Z}^{-k}_G(X) \cong \prod H^{2j}((S^kX)\times_G EG,\mathbb{Z})$, and since $(S^kX_+)\wedge_G EG_+ \cong S^k(X_+\wedge_G EG_+)$, the usual suspension isomorphism implies that $\mathbf{b}\mathfrak{Z}^{-k}_G(X) \cong \prod H^{2j-k}_G(X,\mathbb{Z})$. The module structure is the evident one. $\square$

Similarly, one has the following theorem.

Theorem 4.1.6. *Let X be a compact G-space.*

a. *The zero-th cohomology group $\mathbf{b}\mathfrak{M}^0_G(X)$ is isomorphic to the "units" of $\mathbf{b}\mathfrak{Z}^{\mathbf{0}}_{\mathbf{G}}(\mathbf{X})$. In other words,*

$$\mathbf{b}\mathfrak{M}^0_G(X) \cong 1\times \prod_{j\geq 1} H^{2j}_G(X,\mathbb{Z}) \tag{4.3}$$

under the cup product structure.

b. *If $k>0$ is positive integer, then*

$$\mathbf{b}\mathfrak{M}^{-k}_G(X) \cong \prod_{j\geq 1} H^{2j-k}_G(X,\mathbb{Z}) \tag{4.4}$$

with the usual additive structure.

Proof. **a.** The proof of this assertion follows the same steps of the proof of the corresponding statement in the previous theorem.

b. Write an element $x\in 1\times\prod_{j\geq 1} H^{2j}(S^k(X\times_G EG),\mathbb{Z})$ as a formal sum $1+x_1+x_2+\cdots$, with $x_j\in H^{2j}(S^k(X\times_G EG),\mathbb{Z})$, so that the group operation follows the usual mutiplication law for formal power series. Notice that given $x,y\in 1\times\prod_{j\geq 1} H^{2j}(S^k(X\times_G EG),\mathbb{Z})$ one has $x_i\cup y_j=0$, for all $i,j\geq 1$, since these are singular cohomology classes on the suspension of some space. It follows that $x\cup y = 1+(x_1+y_1)+(x_2+y_2)+\cdots$, and the theorem follows. $\square$

As a corollary one computes the coefficients of the associated Borel theories $\mathbf{b}\mathfrak{Z}^*_G$ and $\mathbf{b}\mathfrak{M}^*_G$, namely, the cohomology groups $\mathbf{b}\mathfrak{Z}^k_G(G/K)$ and $\mathbf{b}\mathfrak{M}^k_G(G/K)$, where $k\in\mathbb{Z}$ and G/K denotes an orbit space for a subgroup $K\subseteq G$. Recall that for a G-space X, one denotes by $\pi^K_n(X)$ the homotopy group $[S^n_K, X]_G$, where S^n_K is the "generalized sphere" $G/K_+\wedge S^n$; cf. [14, I.1]. One can similarly define $\pi^K_n(\mathbf{k}_G)$ for any G-spectrum $\mathbf{k}_G$.

Corollary 4.1.7. a. *Given a subgroup K of the finite group G, there is a ring isomorphism*

$$\mathbf{b}\mathfrak{Z}_G^0(G/K) = \pi_0^K(\mathbf{b}\mathfrak{Z}_G) \cong \prod_{j\geq 0} H^{2j}(BK, \mathbb{Z}),$$

where the multiplicative structure is induced by the cup product on singular cohomology.

b. *For $k > 0$, one has $\mathbf{b}\mathfrak{Z}_G^k(G/K) = 0$ and*

$$\mathbf{b}\mathfrak{Z}_G^{-k}(G/K) = \pi_k^K(\mathbf{b}\mathfrak{Z}_G) \cong \prod_{j\geq 0} H^{2j-k}(BK, \mathbb{Z})$$

as a graded module over $\pi_0^K(\mathbf{b}\mathfrak{Z}_G)$.

c. *The group $\mathbf{b}\mathfrak{M}_G^0(G/K) = \pi_0^K(\mathbf{b}\mathfrak{M}_G)$ is isomorphic to $1 \times \prod_{j\geq 1} H^{2j-k}(BK, \mathbb{Z})$ under the cup product operation. Furthermore, for $k > 0$, one has $\mathbf{b}\mathfrak{M}_G^k(G/K) = 0$ and $\mathbf{b}\mathfrak{M}_G^{-k}(G/K) = \pi_k^M(\mathbf{b}\mathfrak{Z}_G) \cong \prod_{j\geq 1} H^{2j-k}(BK, \mathbb{Z})$ with its usual additive structure.*

The results above show, in particular, that the "units" $1 \times \prod_{j\geq 1} H_G^{2j}(X, \mathbb{Z})$ of the even Borel cohomology of X are the zero-th group of the $R(G)$-graded equivariant cohomology theory $\mathbf{b}\mathfrak{M}_G^*$.

4.2. **Total Chern classes and equivariant Segal questions.** Combining Propositions 3.2.1 and 3.3.3 one obtains a map of $\mathcal{L}(\mathcal{U}_G)$-spaces $B\mathcal{U}_G \to \mathcal{Z}(\mathcal{U}_G)_1$ which in turn induces a map $c_G : \mathfrak{Ku}_G \to \mathfrak{M}_G$ of associated spectra. Passing to Borel counterparts then induces maps of the associated Borel spectra. We collect the various transformations obtained by this construction in the following definition.

Definition 4.2.1. For each finite group G, the map of spectra $c_G : \mathfrak{Ku}_G \to \mathfrak{M}_G$ described above is called the **equivariant total Chern class**. The composition $\mathfrak{Ku}_G \xrightarrow{c_G} \mathfrak{M}_G \xrightarrow{\epsilon_G} \mathbf{b}\mathfrak{M}_G$, denoted by $c_G^{\mathrm{b}} : \mathfrak{Ku}_G \to \mathbf{b}\mathfrak{M}_G$, is called the **total equivariant Borel-Chern class**. We denote by $\hat{c}_G : \mathbf{b}\mathfrak{Ku}_G \to \mathbf{b}\mathfrak{M}_G$ the map induced by c_G. These maps fit into a commutative diagram

$$\begin{tikzcd}
\mathfrak{Ku}_G \arrow[r, "\epsilon'_G"] \arrow[d, "c_G"'] \arrow[dr, "c_G^{\mathrm{b}}"] & \mathbf{b}\mathfrak{Ku}_G \arrow[d, "\hat{c}_G"] \\
\mathfrak{M}_G \arrow[r, "\epsilon_G"'] & \mathbf{b}\mathfrak{M}_G
\end{tikzcd}$$

The next result represents an extension to the equivariant situation of a conjecture of G. Segal [23], which was positively answered in the non-equivariant case in [3]. This result justifies the terminology in Definition 4.2.1.

Theorem 4.2.2. *The zero-th level map $c_G^b : \mathfrak{Ku}_G^0(X) = K_G(X)_o \to \mathbf{b}\mathfrak{M}_G^0(X)$ coincides with the total Chern class map from the equivariant K- theory of X to its Borel cohomology.*

Proof. Let $E \to X$ be an equivariant virtual vector bundle of degree zero over X. The pull back bundle $pr_1^*E \to X \times EG$, under the projection onto first term, descends to a virtual bundle $(pr_1^*E)/G$ over $X \times_G EG$. The total Chern class $c((pr_1^*E)/G) \in 1 \times \prod_{j \geq 1} H^{2j}(X \times_G EG; \mathbb{Z})$ is, by definition, the total Chern class of E in Borel cohomology.

In order to compare $c((pr_1^*E)/G)$ with c_G^{b} one has only to rephrase the previous paragraph in terms of classifying maps into the appropriate spaces, and use the corresponding definitions together with [13]. $\square$

Corollary 4.2.3. *The group of units in even Borel cohomology extends to a generalized equivariant $R(G)$-graded cohomology theory $\mathbf{b}\mathfrak{M}^*_G$, which has the property that the total Chern class map extends to a map of equivariant cohomology theories.*

4.3. **Transfers.** It is a general fact that whenever one has a coherent family of spectra, then the 'transfers' associated to equivariant coverings for the various cohomology theories in the family, are functorialy related; cf. [14, IV.4]. In general, given an equivariant spectrum $\mathbf{k}_G$ and an equivariant covering $E \xrightarrow{\xi} X$, we denote by $\tau_\xi : \mathbf{k}^*_G(E) \longrightarrow \mathbf{k}^*_G(X)$ the associated transfer homomorphism.

We illustrate the general picture with the following "classical" examples. Consider the trivial bundles $\xi : EG \times G/H \longrightarrow EG$ and $\zeta : G/H \longrightarrow \mathrm{pt}$, together with the evident equivariant bundle map $\epsilon : \xi \longrightarrow \zeta$, and let $\theta : BH \longrightarrow BG$ denote the non-equivariant bundle, with fiber G/H obtained as the quotient $\xi/G : BH = EG \times_G G/H \longrightarrow EG/G = BG$.

The following result summarizes the relations among the various transfer homomorphisms associated with these bundles.

Theorem 4.3.1. *Given a subgroup $H \subset G$, one has a commutative diagram*

$$\begin{tikzcd}
 & \mathfrak{M}^0_H(pt) \arrow[rr, "\epsilon^*_H"] \arrow[dd, "\tau_\xi"] & & \mathbf{b}\mathfrak{M}^0_H(pt) \arrow[dd, "\tau_\xi"] \\
R(H)_o = \mathfrak{K}\mathfrak{u}^0_H(pt) \arrow[ur, "c_H"] \arrow[urrr, "c^b_G"] \arrow[rr, "\epsilon'^*_H"'] \arrow[dd, "\mathrm{Ind}^G_H"'] & & \mathfrak{K}\mathfrak{u}^0(BH) \arrow[ur, "\hat{c}_H"'] \arrow[dd, "\tau_\theta"] & \\
 & \mathfrak{M}^0_G(pt) \arrow[rr, "\epsilon^*_G"] & & \mathbf{b}\mathfrak{M}^0_G(pt) \\
R(G)_o = \mathfrak{K}\mathfrak{u}^0_G(pt) \arrow[ur, "c_G"] \arrow[urrr, "c^b_H"] \arrow[rr, "\epsilon'^*_G"'] & & \mathfrak{K}\mathfrak{u}^0(BG) \arrow[ur, "\hat{c}_G"'] &
\end{tikzcd}$$

where the horizontal maps in the front and back faces of the cube are natural maps into respective Borel theories, the vertical maps are transfer homomorphisms and all the other maps are Chern classes. In particular, Ind^G_H *is the induction homomorphim in representation theory and the total Chern class* $c^b_H(\mathrm{Ind}^G_H(V))$ *of an induced representation is given by applying the transfer* τ_ξ *to the Chern class* $c^b_g(V)$ *of* V.

Proof. The groups appearing in the diagram and their relation with the bundles above are explained by observing that if $\mathbf{k}^*_G$ denotes either $\mathfrak{Z}^*_G$ or $\mathfrak{M}^*_G$, and $H \subset G$, then one can identify

$\mathbf{k}^0_G(G/H)$ with $\mathbf{k}^0_H(\mathrm{pt})$. This follows from the coherence properties of our spectra, which give the isomorphisms $\mathbf{k}^0_G(G/H) = [G/H_+, \mathbf{k}_G(0)]_G \cong [S^0, \alpha^\sharp \mathbf{k}_G(0)]_H \cong [S^0, \mathbf{k}_H(0)]_H = \mathbf{k}^0_H(\mathrm{pt})$, where $\alpha : H \hookrightarrow G$ is the inclusion.

The commutativity follows directly from the coherence properties of our spectra. See [14, IV.4.8]. The identification of the transfer in K-theory with Ind_H^G is given in [20]. □

Note that *the transfers above, when restricted to zero-th cohomology groups are multiplicative*, in the sense that they commute with the ordinary cup product. In general, there are several other non-trivial properties which can be inferred from the fact that the transfers come from a generalized cohomology theory. See [20].

4.4. **Analogues in the real case.** All the results established in this chapter can be carried over to the real case by using the constructions and results of section §3.4 together with the results of [10].

The real Chow functor $\mathcal{C}_{\mathbb{R}}$ defined in (3.4.6) has a naïve group completion $\mathcal{Z}_{\mathbb{R}}$. This functor determines an equivariant ring spectrum $\mathfrak{Z}_{\mathbb{R}}$ and a multiplicative spectrum $\mathfrak{M}_{\mathbb{R}}$ in analogy with the complex case above. The underlying non-equivariant spectra are independent of the group G. The zero'th space $\mathfrak{Z}_{\mathbb{R}}(0)$ is non-equivariantly the weak product

$$\prod_{j\geq 0} K(\mathbb{Z}/2\mathbb{Z}, j)$$

with multiplicative structure induced by the cup product (cf. [10]). Furthermore, we have the non-equivariant equivalence $\mathfrak{M}_{\mathbb{R}} \cong \{1\} \times \prod_{j\geq 1} K(\mathbb{Z}/2\mathbb{Z}, j)$ with infinite loop space structure which enhances the cup product. The analogues of Theorems 4.1.4, 4.1.5, 4.1.6 and Corollary 4.1.7 hold for these theories.

The transformation

$$Gr_{\mathbb{R}}(\mathcal{U}_G) \longrightarrow \mathcal{Z}_{\mathbb{R}}(\mathcal{U}_G)$$

of $\mathcal{L}(\mathcal{U}_G)$-spaces induces a map of equivariant spectra

$$w : \mathfrak{ko}_G \longrightarrow \mathfrak{Z}_{\mathbb{R},G}$$

called the **equivariant total Stiefel-Whitney class**. Composition with $\mathfrak{Z}_{\mathbb{R},G} \longrightarrow \mathbf{b}\mathfrak{Z}_{\mathbb{R},G}$ gives an equivariant total Stiefel-Whitney class in the associated Borel theory. In the zero-th level one has an equivariant total Stiefel-Whitney class with values in the Borel cohomology with $\mathbb{Z}/2\mathbb{Z}$-coefficients, which coincides with the classical one. Thus the equivariant version of the Segal question holds for this Borel Stiefel-Whitney class.

The results on transfers also hold in this case.

5. Coefficients

In this chapter we shall compute the coefficients $\mathfrak{Z}^0_G(\mathrm{pt}) = [S^0, \mathcal{Z}(\mathcal{U})]_G = \pi_0(\mathcal{Z}(\mathcal{U})^G)$ of our equivariant theory $\mathfrak{Z}_G$ in the case where G is abelian. Our main result, Theorem 5.2.5, states that in this case the coefficients come from a simple graded ring functor $\mathbb{H}^*(G)$ with particularly nice properties, and they always embed into $\prod_k H^{2k}(G; \mathbb{Z})$. To carry out these computations we use techniques of degeneration by torus actions on Chow varieties.

5.1. **General results for abelian groups.** In this section we derive general results relating $\pi_0(\mathcal{Z}(\mathcal{U})^G)$ of the theory to the representation ring of G, where $\mathcal{U} = \mathcal{U}_G$ stands for our fixed complete G-universe. A key factor in this computation is Theorem 3.3.3 which allows one to work with the Chow monoids $\mathcal{C}(W)$ themselves in order to approximate $\mathcal{Z}(\mathcal{U})$. Our main trick is to use the equivariant degeneration of varieties under action of a maximal complex torus commuting with the action of G.

The first result is a basic reduction formula. Start with a compact abelian (Lie) group G and a finite dimensional G-module W. Fix $p \geq 0$ and denote by $\mathrm{Gr}_p(W)$ the Grassmannian of projective p-planes in $\mathbb{P}(W)$. Let $\mathcal{C}_p(W)$ denote the Chow monoid of effective p-cycles in $\mathbb{P}(W)$. The inclusion $i_W : \mathrm{Gr}_p(W) \longrightarrow \mathcal{C}_p(W)$ is a G-equivariant map and induces a continuous monoid homomorphism $j_W : \mathbb{Z}^+\left\{\mathrm{Gr}_p(W)^G\right\} \longrightarrow \mathcal{C}_p(W)^G$ (where $\mathbb{Z}^+Y = \coprod_d SP^dY$ denotes the free abelian topological monoid on the space Y). Elements of $\mathrm{Gr}_p(W)^G$ are invariant p-planes, and elements of $\mathbb{Z}^+\left\{\mathrm{Gr}_p(W)^G\right\}$ are called **invariant linear p-cycles**.

For each $(p+1)$-dimensional invariant linear subspace $U \subset W$ we consider the submonoid

$$\mathcal{D}_0(U) := \ker (j_U)_* \subset \pi_0\mathbb{Z}^+\left\{\mathrm{Gr}_p(U)^G\right\}$$

where $(j_U)_*$ denotes the induced map on π_0. These are the relations on homotopy classes of sums of G-invariant hyperplanes induced by homotopy through invariant divisors. Let

$$\mathcal{D}(U) \subset \pi_0\mathbb{Z}^+\left\{\mathrm{Gr}_p(W)^G\right\} \tag{5.1}$$

be the image of $\mathcal{D}_0(U)$ induced by the inclusion $U \subset W$.

Theorem 5.1.1. *For any compact abelian group, the map*

$$(j_W)_* : \pi_0\mathbb{Z}^+\left\{Gr_p(W)^G\right\} \longrightarrow \pi_0\left\{\mathcal{C}_p(W)^G\right\}$$

induced by j_W is a monoid surjection whose kernel is generated by the "divisor" relations $\mathcal{D}(U)$ for $U \in Gr_{p+1}(W)^G$.

To prove this result we need a preliminary lemma. Consider $\mathbb{P}^N = \mathbb{P}(\mathbb{C}^{N+1})$ with homogeneous coordinates $[w] = [w_0 : w_1 : \cdots : w_N]$ and the canonical action of the complex N-torus $\mathbb{T} = (\mathbb{C}^\times)^{N+1}/\Delta^\times$ given by

$$\phi_t([w]) = [t_0w_0 : t_1w_1 : \cdots : t_Nw_N], \tag{5.2}$$

where $\Delta^\times$ denotes the diagonal subgroup $(t, t, \ldots, t)$, for $t \in \mathbb{C}^\times$. This action on $\mathbb{P}^N$ induces an action on each component of the Chow monoid $\mathcal{C}_p(\mathbb{P}^N)$ for any $p \geq 0$.

Fix each integer $k \geq 0$ and for each multi-index $I = \{i_0 < i_1 < \cdots < i_p\}$ of length $k+1$ consider the linear subspace

$$\mathbb{C}^I = \{w \in \mathbb{C}^{N+1} : w_i = 0 \text{ if } i \neq i_k \text{ for some } k\}.$$

The subspace $\mathbb{P}^p_I = \mathbb{P}(\mathbb{C}^I)$ is called the I^{th} **coordinate k-plane**. It is clearly $\mathbb{T}$-invariant.

Proposition 5.1.2. *Fix $p \geq 0$ and $d > 0$. Then*

(1) *The fixed-point set of* $\mathbb{T}$ *acting on* $\mathcal{C}_{p,d}(\mathbb{P}^N)$ *is finite and consists of coordinate linear cycles* $\sum_I n_I \mathbb{P}^p_I$ *where each* $n_I \in \mathbb{Z}^+$ *and* $\sum n_I = d$.

(2) *Let* σ *be an irreducible,* $\mathbb{T}$*-fixed algebraic curve on* $\mathcal{C}_{p,d}(\mathbb{P}^N)$. *Then an open, dense subset of* σ *can be written as a parameterized curve of the form*

$$\sigma(t) \;=\; \sum_i n_i \sigma_i(t) \qquad\qquad t \in S$$

where S *is an irreducible algebraic curve, and where for each* i, $n_i \in \mathbb{Z}$ *and*

$$\bigcup_t |\sigma_i(t)| \;\subseteq\; \mathbb{P}^{p+1}_I$$

for some coordinate $(p+1)$*-plane* $\mathbb{P}^{p+1}_I$. *In particular, each cycle* $\sigma_i(t)$ *is a divisor in* $\mathbb{P}^{p+1}_I$.

Proof. Let $\mathbb{L} \cong \mathbb{C}^N$ denote the Lie algebra of $\mathbb{T}$ and for each $x \in \mathbb{P}^N$ let $\mathbb{L}(x) \subset T_x(\mathbb{P}^N)$ denote the tangent space to the $\mathbb{T}$-orbit $\mathbb{T}(x)$ at x. Set

$$\Sigma_p \;=\; \{x \in \mathbb{P}^N \;:\; \dim_{\mathbb{C}} \mathbb{L}(x) \le p\}.$$

then it is straightforward to see that

$$\Sigma_p \;=\; \bigcup_I \mathbb{P}^p_I. \tag{5.3}$$

Now if $c \in \mathcal{C}_{p,d}(\mathbb{P}^N)$ is $\mathbb{T}$-fixed, then its support $|c|$ is a $\mathbb{T}$-invariant algebraic subset of dimension p. Let V be an irreducible component of $|c|$ and fix any $x \in \mathrm{Reg}(V)$. Then V is $\mathbb{T}$-invariant and so

$$\mathbb{L}(x) \subset T_x V.$$

It follows (since $\dim(T_x V) = p$) that $x \in \Sigma_p$. We conclude that $|c| \subset \Sigma_p$ and (1) now follows from (5.3).

For (2) we begin by observing that the subvariety

$$W := \bigcup_{c \in \sigma} |c| \;\subset \mathbb{P}^N$$

is $\mathbb{T}$-invariant and of dimension $p+1$. It therefore follows as above that

$$W \subset \Sigma_{p+1} = \bigcup_I \mathbb{P}^{p+1}_I. \tag{5.4}$$

Now the irreducibility of σ together with the finiteness of the fixed-point set of $\mathbb{T}$ on $\mathcal{C}_{p,d}(\mathbb{P}^N)$ imply that σ is the closure of the $\mathbb{T}$-orbit of a generic point $\mathbf{c}$ in its support. Thus W is the closure of the set

$$W^0 := \bigcup_{t \in \mathbb{T}/\mathbb{T}_{\mathbf{c}}} |t_* \mathbf{c}| \;\subset\; \bigcup_{\mathbf{I}} \mathbb{P}^{\mathbf{p+1}}_{\mathbf{I}},$$

where $\mathbb{T}_{\mathbf{c}}$ is the isotropy subgroup of $\mathbf{c}$. We set $S := \mathbb{T}/\mathbb{T}_{\mathbf{c}}$ and note that S embeds as an open dense subset of σ (as the $\mathbb{T}$-orbit of $\mathbf{c}$). Consequently σ can be written generically as the parameterized curve $\sigma(t) = t_*c$ for $t \in S$. Write $c = \sum_i n_i V_i$ where the n_i are positive integers and the V_i are irreducible p-dimensional subvarieties of $\mathbb{P}^N$. Then

$$\sigma(t) \;=\; t_*c \;=\; \sum n_i t_* V_i \;\equiv\; \sum_i n_i \sigma_i(t) \qquad \text{for} \qquad t \in S.$$

Each V_i is contained in some $\mathbb{P}^{p+1}_I$ by (5.4), and so the curve σ_i is also contained in $\mathbb{P}^{p+1}_I$. $\square$

Proof. [of Theorem 5.1.1] Choose homogeneous coordinates $[w] = [w_0 : w_1 : \cdots : w_N]$ so that each coordinate line is G-invariant. (This is possible since the irreducible representations of G are all 1-dimensional.) Let $\mathbb{T}$ be the complex N-torus acting on $\mathbb{P}(W)$ via this coordinate decomposition as in (5.2). Note that *the actions of G and $\mathbb{T}$ commute*. It follows for example that $\mathbb{T}$ preserves the components of $\mathbb{Z}^+(\mathrm{Gr}_p(W))^G$ and $\mathcal{C}_p(W)^G$.

We first prove surjectivity. Fix $c \in \mathcal{C}_p(W)^G$. It is a basic result (cf. [2]) that the closure of this orbit $\overline{\mathbb{T}(c)}$ contains $\mathbb{T}$-fixed points. (This result is usually stated for $\mathbb{T}$-actions on smooth varieties. However each component of $\mathcal{C}_p(W)$ admits a $\mathbb{T}$-equivariant embedding into some projective space $\mathbb{P}^M$. In fact the standard Chow embedding is $\mathbb{T}$-equivariant. One can then ignore $\mathcal{C}_p(W)$ and apply the theorem to $\mathbb{T}$-orbits in $\mathbb{P}^M$.)

Now by Proposition 5.1.2(1) every $\mathbb{T}$-fixed cycle is a sum of G-invariant coordinate planes. This proves the surjectivity in Theorem 5.1.1.

We now prove the second assertion. Suppose $a, b \in \mathbb{Z}^+(\mathrm{Gr}(W)^G)$ lie in the same connected component of $\mathcal{C}_{p,d}(W)^G$ under the inclusion j_W. Then they can be joined in $\mathcal{C}_{p,d}(W)^G$ by a connected *algebraic* curve σ. Consider σ as a 1-cycle in $\mathcal{C}_{p,d}(W)^G$, and hence as a point in some Chow variety $\mathcal{C}_1(\mathcal{C}_{p,d}(W)^G) \subset \mathcal{C}_1(\mathbb{P}^M)$. Choose a fixed-point σ_0 in the orbit closure $\overline{\mathbb{T}(\sigma)}$ of σ in $\mathcal{C}_1(\mathbb{P}^M)$. The point σ_0 corresponds to a $\mathbb{T}$-fixed 1-cycle in $\mathbb{P}^M$. Now, since $\mathbb{T}$ preserves the subvariety $\mathcal{C}_{p,d}(W)^G \subset \mathbb{P}^M$, we have $|t_*\sigma| \subset \mathcal{C}_{p,d}(W)^G$ for all $t \in \mathbb{T}$, and therefore $|\sigma_0| \subset \mathcal{C}_{p,d}(W)^G$. Since each curve $t_*\sigma$ is connected, so is σ_0, and since a and b are $\mathbb{T}$-fixed, we have $a, b \in |\sigma_0|$. Thus $|\sigma_0|$ is a connected, $\mathbb{T}$-invariant algebraic curve in $\mathcal{C}_{p,d}(W)^G$ joining a to b. Each irreducible component of σ_0 is $\mathbb{T}$-invariant. Furthermore, any point in the intersection of two distinct components must be a $\mathbb{T}$-fixed point, i.e., an element of $\mathbb{Z}^+(\mathrm{Gr}_p(W)^G)$ by Proposition 5.1.2(1). The result now follows from Proposition 5.1.2(2) $\square$

Suppose now that W is a complex linear G-space of dimension w. Then we can apply Theorem 5.1.1 to the map

$$\mathrm{Gr}(W) {:=} \mathrm{Gr}_{w-1}(W \oplus W) \;\longrightarrow\; \mathcal{C}_{w-1}(W \oplus W) {:=} \mathcal{C}(W) \tag{5.5}$$

Let $\mathcal{Z}(W) = \mathcal{Z}_{w-1}(W \oplus W)$ denote the naïve group completion of $\mathcal{C}(W)$. Note that for any space Y there is an isomorphism $\mathbb{Z}^+\pi_0(Y^G) \cong \pi_0\mathbb{Z}^+(Y^G)$. Therefore taking group completions in (5.1) gives us the following.

Corollary 5.1.3. *For any invariant subspace $W \subset \mathcal{U}$ the natural inclusion induces a surjective homomorphism of abelian groups*

$$\mathbb{Z}\pi_0 \left\{ Gr(W)^G \right\} \to \pi_0 \left\{ \mathcal{Z}(W)^G \right\}$$

whose kernel is generated by the "divisor" relations $\mathcal{D}(U)$ for $U \in Gr_w(W \oplus W)^G$.

We now consider the direct limits

$$B\mathcal{U} := \varinjlim_{W \subset \mathcal{U}} \mathrm{Gr}(W) \qquad \text{and} \qquad \mathcal{Z}\mathcal{U} := \varinjlim_{W \subset \mathcal{U}} \mathcal{Z}(W) \tag{5.6}$$

and recall that under the join pairing, $\mathcal{Z}\mathcal{U}$ and $\mathcal{Z}\mathcal{U}^G$ are homotopy-commutative rings. Corollary 5.1.3 immediately gives the following result.

Theorem 5.1.4. *For any compact abelian group G, the inclusion* (5.5) *induces a surjective ring homomorphism*

$$\mathbb{Z}\pi_0 \left\{ B\mathcal{U}^G \right\} \longrightarrow \pi_0 \left\{ \mathcal{Z}\mathcal{U}^G \right\}$$

whose kernel is generated by the images of the "divisor" relations $\mathcal{D}(U)$ in the direct limit.

5.2. **Computations for finite groups.** In this section we shall explicitly compute $\pi_0 \left\{ \mathcal{Z}\mathcal{U}^G \right\}$ for finite G. The main point is to compute the relations $\mathcal{D}(\mathcal{U})$ given by invariant divisors; cf. (5.1). We set the stage by recalculating $\pi_0(B\mathcal{U}^G)$ in a specific manner which is subsequently generalized to cycles. This approach is similar to calculations in [22], and its relevance here lies on its extension to arbitrary cycles.

Let G be a finite (not necessarily abelian) group of order γ and let $B\mathcal{U}$ be defined as in (5.6). We want to compute

$$\pi_0(B\mathcal{U}^G) = \pi_0 \left\{ \varinjlim_{W \subset \mathcal{U}} \mathrm{Gr}(W)^G \right\}.$$

Let $\xi_1, \ldots, \xi_m$ denote the distinct irreducible complex representations of G and set $|\xi_i| = \dim \xi_i$. For each invariant w-plane $P \subset W \oplus W$ there is a unique G-equivalence $P \cong \bigoplus \xi_i^{\oplus a_i}$ where $\sum a_i |\xi_i| = w$. Furthermore, P is homotopic to $P' \cong \bigoplus \xi_i^{\oplus a_i'}$ through G-invariant planes if and only if $a_i = a_i'$ for all i. (this follows from the continuity of characters.) Thus if $W \cong \bigoplus \xi_i^{\oplus n_i}$ we see that

$$\pi_0 \left\{ \mathrm{Gr}(W)^G \right\} \cong \left\{ (a_1, \ldots, a_m) \in \mathbb{Z}^m \ : \ 0 \le a_i \le 2n_i \text{ and } \sum a_i |\xi_i| = w \right\}.$$

We now set

$$V = \xi_1 \oplus \cdots \oplus \xi_m$$

and consider the cofinal family $V^{\oplus n} = V \oplus \cdots \oplus V$ in $\mathcal{U}$. Then

$$\begin{aligned} \pi_0 \left\{ \mathrm{Gr}(V^{\oplus n})^G \right\} &\cong F_n \\ &:= \left\{ (a_1, \ldots, a_m) \in \mathbb{Z}^m \ : \ 0 \le a_i \le 2n \text{ and } \sum a_i |\xi_i| = nv \right\} \end{aligned}$$

where $v = \dim V = \sum |\xi_i|$. Since $\mathcal{U}$ is a limit of $V^{\oplus n} \oplus V^{\oplus n}$, we see that $\pi_0(B\mathcal{U}^G) = \varinjlim F_n$ under the family of inclusion maps $\cdots \subset F_n \subset F_{n+1} \subset \dots$ given by

$$(a_1, \dots, a_m) \ \mapsto \ (a_i + 1, \dots, a_m + 1), \tag{5.7}$$

since at each stage we sent $P \in \mathrm{Gr}(V^{\oplus n})$ to $P \oplus (V \oplus \{0\}) \cong P + (1, 1, \cdots, 1) \in \mathrm{Gr}(V^{\oplus n})$. We now write $R(G)$ as the free abelian group with generators $\xi_1, \cdots, \xi_m$, and we write this **multiplicatively** as in 3.2.7. Recall the subgroup

$$R(G)_0 \ = \ \left\{\xi_1^{a_1} \dots \xi_m^{a_m} \in R(G) \ : \ \sum a_i \dim \xi_i = 0\right\}.$$

of representations of virtual dimension zero. Then we have the following; cf. (3.9).

Proposition 5.2.1. $$R(G)_0 \ \cong \ \varinjlim F_n \ = \ \pi_0\left(B\mathcal{U}^G\right).$$

Proof. The map $F_n \hookrightarrow R(G)_0$ given by

$$(a_1, \cdots, a_m) \ \mapsto \ \prod_{i=1}^{m} \xi_i^{(a_i - n)}$$

is compatible with the inclusions (5.7) and therefore induce a mapping $\varinjlim F_n \to R(G)_0$ which is easily seen to be an isomorphism. □

By Proposition 5.2.1 we have

$$\begin{aligned} \mathbb{Z}\pi_0\left\{(B\mathcal{U})^G\right\} &= \mathbb{Z}\left[R(G)_0\right] \\ &= \mathbb{Z}\left[\xi_1, \cdots, \xi_m, \xi_1^{-1}, \cdots, \xi_m^{-1}\right]_0 \\ &:= \mathbb{Z}\left[\xi, \xi^{-1}\right]_0 \end{aligned} \tag{5.8}$$

where $\mathbb{Z}\left[\xi, \xi^{-1}\right]_0$ denotes the ring of all Laurent polynomials in $\xi_1, \cdots, \xi_m$ of degree zero, i.e., integral linear combinations on monomials $\xi_1^{a_1} \dots \xi_m^{a_m}$ with $\sum a_i = 0$.

We now assume that G is abelian so that $m = \gamma$. For convenience we shall reindex the irreducible representations by ξ_ρ for $\rho \in \widehat{G}$ where $\widehat{G} \cong G$ denotes the **character group** of G. Thus we have

$$V \ = \ \bigoplus_{\rho \in \widehat{G}} \xi_\rho.$$

Fix now an element $\vec{a} \ = \ \{a_\rho\}_{\rho \in \widehat{G}} \ \in \ \mathbb{Z}^+\widehat{G}$, and consider the corresponding invariant subspace

$$U = U(\vec{a}) = \bigoplus_{\rho} \xi_\rho^{\oplus a_\rho} \subset V^{\oplus n} \oplus V^{\oplus n} \qquad \text{where}$$

$$\dim(U) = n\gamma + 1 = \sum a_\rho$$

For fixed d let $\mathrm{Div}_d(U)$ denote the set of divisors of degree d on $\mathbb{P}(U)$. Then $\mathrm{Div}_d(U) = \mathbb{P}(S^d(U))$ where $S^d(U)$ is the d^{th} symmetric tensor power of U. Note that $\mathrm{Div}_d(U)$ is a

G-space with fixed-point set

$$\mathrm{Div}_d(U)^G = \coprod_{\rho \in \widehat{G}} \mathbb{P}\left(S^d(U)_\rho\right) \tag{5.9}$$

where $S^d(U)_\rho \subset S^d(U)$ denotes the G-invariant subspace with character ρ. Observe also that for $\sigma \in \widehat{G}$,

$$S^d(U)_\sigma \;=\; \bigoplus_{\substack{\sum d_\rho = d \\ \sum d_\rho \rho = \sigma}} \left\{ \bigotimes_{\rho \in \widehat{G}} S^{d_\rho}\left(\xi_\rho^{\oplus a_\rho}\right) \right\}. \tag{5.10}$$

Now the coordinate linear divisors lying in $\bigotimes_\rho S^{d_\rho}\left(\xi_\rho^{\oplus a_\rho}\right)$ correspond to monomials $\prod_\rho t_\rho^{d_\rho}$ where t_ρ is the coordinate on ξ_ρ. This divisor on $\mathbb{P}(U(\vec{a}))$ can be written as the linear sum of hyperplanes $\sum d_\rho \ \mathrm{Div}(t_\rho) = \sum d_\rho \mathbb{P}(U(\vec{a} - \vec{e_\rho}))$ where $\vec{e_\rho} = \{\delta_{\rho\rho'}\}_{\rho'}$ and where $\delta_{\rho\rho'} = 1$ if $\rho = \rho'$ and 0 otherwise. In the notation of (5.8) this is the coordinate linear cycle

$$\sum_\rho d_\rho \prod_\sigma \xi_\sigma^{(a_\sigma - \delta_{\sigma\rho})} = \Big(\prod_\sigma \xi_\sigma^{a_\sigma}\Big)\Big(\sum_\rho d_\rho \xi_\rho^{-1}\Big).$$

By (5.9) we see that two such cycles are equivalent in $\pi_0\{\mathcal{CU}^G\}$ if they correspond to the same invariant subspace $S^d(U)_\sigma$. By (5.10) this means that

$$\xi^{\vec{a}} \sum_\rho d_\rho \xi_\rho^{-1} \equiv \xi^{\vec{a}} \sum_\rho d'_\rho \xi_\rho^{-1} \ \text{ in } \ \pi_0\{\mathcal{CU}\} \qquad \text{if} \qquad \begin{cases} \sum_\rho d_\rho = \sum_\rho d'_\rho & \text{and} \\ \sum_\rho d_\rho \rho = \sum_\rho d'_\rho \rho \end{cases} \tag{5.11}$$

where $\xi^{\vec{a}} \equiv \prod_\sigma \xi_\sigma^{a_\sigma}$. We now pass to the group completion $\mathcal{ZU}$ of $\mathcal{CU}$. From (5.11) and Theorem 5.1.4 it is straightforward to calculate the following.

Lemma 5.2.2. *The homomorphism $\mathbb{Z}\pi_0\{B\mathcal{U}^G\} \longrightarrow \pi_0\{\mathcal{ZU}^G\}$ descends to an isomorphism*

$$\mathbb{Z}[\xi, \xi^{-1}]_0/\mathcal{D} \cong \ \pi_0\{\mathcal{ZU}^G\}$$

where $\mathcal{D}$ is the ideal generated by the elements

$$\xi_1 \sum_\rho d_\rho \xi_\rho^{-1} \qquad \textit{for} \quad \begin{cases} \sum_\rho d_\rho = 0 & \textit{and} \\ \sum_\rho d_\rho \rho = 0 \end{cases}.$$

We now make a change of variables by setting $\zeta_\rho := \xi_r/\xi_1$ for $\rho \in \widehat{G}$. Note that $\zeta_1 = 1$. This change of variables induces an isomorphism

$$\mathbb{Z}[\xi, \xi^{-1}]_0 \ \cong \ \mathbb{Z}[\zeta, \zeta^{-1}].$$

The isomorphism of Lemma 5.2.2 now becomes the isomorphism

$$\pi_0\{\mathcal{ZU}^G\} \ \cong \ \mathbb{Z}[\zeta, \zeta^{-1}]/\mathcal{D} \tag{5.12}$$

where $\mathcal{D}$ is the ideal generated by the elements

$$\sum_\rho d_\rho \zeta_\rho^{-1} \qquad \text{for} \quad \begin{cases} \sum_\rho d_\rho = 0 & \text{and} \\ \sum_\rho d_\rho \rho = 0 & \end{cases}. \tag{5.13}$$

Note that the ring $\pi_0\{\mathcal{ZU}^G\}$ is graded by the **augmentation** deg : $\pi_0\{\mathcal{ZU}^G\} \to \mathbb{Z}$, corresponding to projective degree, which sends $\sum d_\alpha \zeta^\alpha$ to $\sum d_\alpha$.

We now make a further change of variables by setting

$$\zeta_\rho^{-1} := 1 + x_\rho \qquad \text{and} \qquad \zeta_\rho := 1 + \bar{x}_\rho$$

with the convention that $x_1 = \bar{x}_1 = 0$. The element x_ρ is called the **first Chern class** of the representation $\rho \in \widehat{G}$ in $\pi_0\{\mathcal{ZU}^G\}$.

Proposition 5.2.3. *In the variables x_ρ, $\bar{x}_\rho$ the isomorphism of Lemma 5.2.2 becomes*

$$\pi_0\{\mathcal{ZU}^G\} \;\cong\; \mathbb{Z}[x, \bar{x}]/\mathcal{D}$$

where $\mathcal{D}$ is the ideal generated by the relations $(1 + x_\rho)(1 + \bar{x}_\rho) = 1$ for $\rho \in \widehat{G}$ and

$$\sum_\rho d_\rho x_\rho = 0 \qquad \textit{whenever} \quad \sum_\rho d_\rho \rho = 0 \quad \textit{in} \quad \widehat{G}.$$

Proof. The first set of relations reflect the fact that $\zeta_\rho \zeta_\rho^{-1} = 1$. The second is deduced easily from (5.13) ☐

Definition 5.2.4. Let $\mathbb{H}^*(G) \subset \mathbb{Z}[x, \bar{x}]/\mathcal{D} \cong \pi_0\{\mathcal{ZU}^G\}$ denote the subring with unit generated by the classes x_ρ. Note that $\mathbb{H}^*(G) = \mathbb{Z}[x]/\mathcal{D}_0$ where $\mathcal{D}_0$ is the homogeneous ideal

$$\mathcal{D}_0 := \left\langle \sum d_\rho x_\rho \mid \sum d_\rho \rho = 0 \text{ in } \widehat{G} \right\rangle. \tag{5.14}$$

Note that $\mathbb{H}^*(G)$ inherits a natural grading from the usual one on $\mathbb{Z}[x]$.

Theorem 5.2.5. *$\mathbb{H}^*(G)$ is a graded ring functor on the category of finite abelian groups with the following properties.*

1. *$\mathbb{H}^0(G) \cong \mathbb{Z}$ and $\mathbb{H}^1(G) \cong \widehat{G}$.*
2. *$\mathbb{H}^*(G \oplus G') = \mathbb{H}^*(G) \otimes \mathbb{H}^*(G')$.*
3. *If G is cyclic, then there exists a graded ring isomorphism*

$$\mathbb{H}^*(G) \;\cong\; H^{2*}(G; \mathbb{Z})$$

with the cohomology of G with coefficients in the trivial G- module $\mathbb{Z}$.

For any finite abelian group G there is a natural isomorphism

$$\pi_0\{\mathcal{ZU}^G\} \;\cong\; \mathbb{H}^*(G)_{\mathcal{M}} \tag{5.15}$$

where $\mathcal{M} \subset \mathbb{H}^*(\mathcal{G})$ is the multiplicative system generated by the total Chern classes $\{1 + x_\rho\}_{\rho \in \widehat{G}}$ of the irreducible representations of G.

Proof. For (1) consider the homomorphism $\phi : \mathbb{Z}\langle x_\rho \rangle_{\rho\neq 1} \to \widehat{G}$ given by $\phi(\sum d_\rho x_\rho) = \sum d_\rho \rho$. By (5.14) this evidently descends to the desired isomorphism. To prove (3), suppose $G = \mathbb{Z}/m\mathbb{Z}$ and let $x_1, \dots, x_{m-1}$ be the generators of $\mathbb{Z}[x]$. Then $\mathcal{D}_0$ is generated by the relations $x_d = dx_1$ for $d = 2, \dots, m-1$. Thus setting $x = x_1$, one sees directly that $\mathbb{H}^*(\mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}[x]\langle mx\rangle \cong H^{2*}(\mathbb{Z}/m\mathbb{Z}; \mathbb{Z})$.

For (2) we write $\mathbb{H}^*(G) = \mathbb{Z}[x]/\mathcal{D}_0$ and $\mathbb{H}^*(G') = \mathbb{Z}[y]/\mathcal{D}'_0$ as in (5.2.4). Then $\mathbb{H}^*(G \oplus G') = \mathbb{Z}[x,y]/\mathcal{E}_\prime$ where

$$\begin{aligned}\mathcal{E}_\prime &= \left\langle \sum d_\rho x_\rho + \sum d'_{\rho'} y_{\rho'} \mid \sum d_\rho \rho + \sum d'_{\rho'}\rho' = 0 \text{ in } \widehat{G} \oplus \widehat{G}' \right\rangle \\ &= \left\langle \sum d_\rho x_\rho \mid \sum d_\rho \rho = 0 \text{ in } \widehat{G} \right\rangle + \left\langle \sum d'_{\rho'} y_{\rho'} \mid \sum d'_{\rho'}\rho' = 0 \text{ in } \widehat{G}' \right\rangle = \mathcal{D}_0 + \mathcal{D}'_0.\end{aligned}$$

This proves the first part of the theorem. The second part follows from Proposition 5.2.3. $\square$

Note 5.2.6. For cyclic groups (in fact for products of cyclic groups whose orders are pairwise coprime) the functors $\mathbb{H}(G)$ and $H(G;\mathbb{Z})$ coincide, but in general $\mathbb{H}(G)$ is simpler. There is however a canonical embedding of graded rings $\mathbb{H}^*(G) \subset H^{2*}(G;\mathbb{Z})$.

Note 5.2.7. Let $\widehat{\mathbb{H}}^*(G) = \prod_k \mathbb{H}^k(G)$ and $\widehat{H}^{2*}(G;\mathbb{Z}) = \prod_k H^{2k}(G;\mathbb{Z})$ denote the completions of $\mathbb{H}^*(G)$ and $H^{2*}(G;\mathbb{Z})$ with respect to the augmentation ideals. Then there are natural embeddings

$$\mathbb{H}^*(G)_{\mathcal{M}} \subset \widehat{\mathbb{H}}^*(G) \subset \widehat{H}^{2*}(G;\mathbb{Z}) \tag{5.16}$$

Combining (5.14) and (5.15) gives an embedding $\pi_0\left\{\mathcal{Z}\mathcal{U}^G\right\} \subset \widehat{H}^{2*}(G;\mathbb{Z})$ which is parallel to the fact that non-equivariantly we have $\mathcal{Z}\mathcal{U} \cong \prod_j K(\mathbb{Z}, 2j)$ and so $\mathcal{Z}\mathcal{U}$ classifies the functor $\widehat{H}^{2*}(\bullet;\mathbb{Z})$

5.3. **The general case.** The calculations above indicate that $\pi_0\{\mathcal{Z}\mathcal{U}\}$ might be a useful functor for finite groups, and it would be interesting to compute it in some non-abelian cases.

The higher coefficients $\pi_k\{\mathcal{Z}\mathcal{U}\}$ (and more generally $\pi^V\{\mathcal{Z}\mathcal{U}\}$ for representations V of G) are also interesting and there are some tools in place for trying to compute them. In particular, there is an Equivariant Algebraic Suspension Theorem proved in [12] which asserts the following. Let G be a finite group of order γ, and suppose V is a complex G–module. Let V_0 denote the regular representation of G. Then for any $p < \dim V$, there is a sequence of continuous G-homomorphisms

$$\begin{aligned}&\mathcal{Z}_p(\mathbb{P}(V)) \xrightarrow{\Sigma^{V_0}} \mathcal{Z}_{p+\gamma}(\mathbb{P}(V \oplus V_0)) \xrightarrow{\Sigma^{V_0}} \dots \xrightarrow{\Sigma^{V_0}} \\ &\qquad \mathcal{Z}_{p+n\gamma}(\mathbb{P}(V \oplus V_0^n)) \xrightarrow{\Sigma^{V_0}} \mathcal{Z}_{p+(n+1)\gamma}(\mathbb{P}(V \oplus V_0^{n+1})) \xrightarrow{\Sigma^{V_0}} \dots\end{aligned} \tag{5.17}$$

which associate to a cycle c the join $\Sigma^{V_0}(c) = \mathbb{P}(V_0)\sharp c$. The theorem asserts that there is an integer n, depending explicitly on V and p, so that after the n^{th} iteration, all maps in (5.17) are G-homotopy equivalences.

If these maps were G-homotopy equivalences from the first iteration, then we would be able to compute all higher coefficients in the theory, since this computation would be reduced to the equivariant Dold-thom Theorem established in [15]. However, this is not true. The integers n in the Suspension Theorem are nearly sharp as shown by many examples.

6. Embellishments and open problems

We now examine some extensions of the results developed above. We begin by showing that our constructions can be carried out meaningfully on any algebraic variety. Thereby, one enhances the total Chern class in morphic cohomology developed in [6]

6.1. **Applications to algebraic varieties.** The basic constructions of Chapter 3 can be carried into algebraic geometry as follows. Let X be a projective algebraic variety of dimension n with a finite group of automorphisms G. Fix a complete G-universe $\mathcal{U}$. Then to each G-module $V \subset \mathcal{U}$ we associate the abelian topological monoid

$$\mathcal{C}_X(V) := \mathrm{Mor}(X, \mathcal{C}(V))$$

where $\mathcal{C}(V)$ is Chow monoid defined in (3.1); cf. [6] and [7]. The mappings $C(f)$ defined in (3.2), and the biadditive join pairing defined in (3.3) extend in an obvious way to these monoids, and the arguments of §3.1 carry over directly to prove that $(\mathcal{C}_X, \sharp)$ is a $G\mathcal{I}_*$-functor.

Let $\mathcal{Z}_X(V)$ denote the naïve topological group completion of the monoid $\mathcal{C}_X(V)$. It is proved in [7] that $\mathcal{Z}_X(V)$ is (non-equivariantly) homotopy equivalent to $\Omega B\mathcal{C}_X(V)$. Just as above we see that $(\mathcal{Z}_X, \sharp)$ is a $G\mathcal{I}_*$-functor, and the obvious inclusion $(\mathcal{C}_X, \sharp) \subset (\mathcal{Z}_X, \sharp)$ yields a transformation of $G\mathcal{I}_*$- functors. Therefore, passing to the direct limits gives a map

$$\mathcal{C}_X(\mathcal{U}) \longrightarrow \mathcal{Z}_X(\mathcal{U})$$

of $\mathcal{L}(\mathcal{U})$-spaces which can be shown to be an additive G-group completion as in 3.3.3.

Definition 6.1.1. The G-ring spectrum $\mathfrak{Z}_X$ associated to the $\mathcal{L}(\mathcal{U})$-space $\mathcal{Z}_X(\mathcal{U})$ is called the **morphic spectrum of** X.

We denote by $\mathfrak{M}_X$ the associated **multiplicative morphic spectrum** with the property that the map $\mathcal{Z}_X(\mathcal{U})_1 \longrightarrow \mathfrak{M}_X(0)$ is a G-group completion with respect to the cup product.

When $G = e$, the spectrum $\mathfrak{Z}_X$ corresponds to the morphic cohomology of X introduced in [6]. In the notation of [6] we have

$$\mathcal{Z}_X(\mathcal{U}) = \lim_{q\to\infty} \mathcal{Z}^q(X; \mathbb{P}^q) = \mathcal{Z}^\infty(X; \mathbb{P}^\infty) \cong \prod_{s\geq 0} \mathcal{Z}^s(X)$$

and therefore

$$\pi_i \mathcal{Z}_X(\mathcal{U}) = \bigoplus_{s\geq 0} L^s H^{2s-i}(X)$$

(cf. [6, Thm 2.10]). This direct sum is finite since $L^sH^k(X) = 0$ whenever $k > 2n$ (see [6, 9.8]). Furthermore, if X is smooth, then $L^sH^{2s-i}(X) \cong H^{2s-i}(X;\mathbb{Z})$ whenever $s \geq n$ by [7]. When X is smooth we also have that

$$\pi_0\mathcal{Z}_X(\mathcal{U}) \;=\; \bigoplus_{s\geq 0} L^sH^{2s}(X) \;\cong\; \mathcal{A}_0 \times \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n$$

where $\mathcal{A}_p$ denotes the group of algebraic p-cycles on X modulo algebraic equivalence (see [7, Thm 5.1]). This shows that in general $\mathcal{Z}_X(\mathcal{U})$ depends strongly on the algebraic structure of X.

Note that by the above $\pi_0\mathcal{Z}_X(\mathcal{U})$ is a finitely graded, commutative ring with unit. It follows that when $G = e$, the space $\mathcal{Z}_X(\mathcal{U})_1$ is group-like with respect to the cup-product. Thus, in this case, there is a homotopy equivalence $\mathfrak{M}_X(0) \cong \mathcal{Z}_X(\mathcal{U})_1$. If X is smooth, then by duality [7] we have $\pi_0\mathfrak{M}_X(0) \cong \{1\} \times \mathcal{A}_{n-1} \times \cdots \times \mathcal{A}_0$.

From the discussion above we see that the coefficients in the theory $\mathfrak{Z}_X$ could be justifiably called the **equivariant morphic cohomology of** X.

We now replace $\mathcal{C}_X(V)$ with the $G\mathcal{I}_*$-subspace

$$Gr_X(V) := \mathrm{Mor}(X, Gr(V)),$$

and note that the direct limit $Gr_X(\mathcal{U})$ is a $\mathcal{L}(\mathcal{U})$-space.

Definition 6.1.2. The G-spectrum $\mathfrak{Ku}_X$ associated to the $\mathcal{L}(\mathcal{U})$-space $Gr_X(\mathcal{U})$ is called the **Grassmann spectrum of** X.

In the language of [6] we have that $Gr_X(V) = \mathbf{Vect}^v(X)$, and it is shown there that $\pi_0 Gr_X(V)$ coincides with algebraic equivalence classes of algebraic vector bundles of rank v on X which are generated by their global sections. Thus $Gr_X(\mathcal{U})$ is the additive monoid of stable equivalence classes of such bundles under Whitney sum. The map $Gr_X(\mathcal{U}) \to \mathfrak{Ku}_X(0)$ into the zero'th space of the spectrum is a G-group completion. In particular we see that (non-equivariantly) $\pi_0\mathfrak{Ku}_X(0)$ represents the group completion $K_{\mathrm{alg}}(X)_o$ of the monoid of equivalence classes of algebraic vector bundles generated by their global sections.

Observe now that the inclusion

$$(Gr_X, \oplus) \longrightarrow (\mathcal{C}_X, \sharp)$$

is a transformation of $G\mathcal{I}_*$-functors, which induces a map

$$Gr_X(\mathcal{U}) \to \mathcal{Z}_X(\mathcal{U})_1$$

of $\mathcal{L}(\mathcal{U})$-spaces. This in turn induces a map

$$\mathbf{c} : \mathfrak{Ku}_X \;\longrightarrow\; \mathfrak{M}_X$$

of G-spectra, called the **total Chern class in equivariant morphic cohomology**

It is interesting to examine the "continuous analogue" of these constructions given by the transformation $\mathrm{Map}(X, Gr(V)) \;\longrightarrow\; \mathrm{Map}(X, \mathcal{Z}(V))$ of $G\mathcal{I}_*$-functors. Here we use the notation $\mathrm{Map}(X,Y)$ instead of $F(X,Y)$, for compatibility with [6]. Note that there is a commutative diagram of $G\mathcal{I}_*$-transformations

$$\begin{array}{ccc} \mathrm{Mor}(X, Gr(V)) & \longrightarrow & \mathrm{Mor}(X, \mathcal{C}(V)) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Map}(X, Gr(V)) & \longrightarrow & \mathrm{Map}(X, \mathcal{Z}(V)) \end{array}$$

leading to a commutative diagram of $\mathcal{L}(\mathcal{U})$-spaces

$$\begin{array}{ccc} Gr_X(\mathcal{U}) & \longrightarrow & \mathcal{Z}_X(\mathcal{U})_1 \\ \big\downarrow & & \big\downarrow \\ \mathrm{Map}(X, Gr(\mathcal{U})) & \longrightarrow & \mathrm{Map}(X, \mathcal{Z}(\mathcal{U})_1) \end{array}$$

and therefore a diagram of G-spectra

$$\begin{array}{ccc} \mathfrak{Ku}_X & \longrightarrow & \mathfrak{M}_X \\ \big\downarrow & & \big\downarrow \\ \mathrm{Map}(X, \mathfrak{Ku}) & \longrightarrow & \mathrm{Map}(X, \mathfrak{M}). \end{array}$$

Suppose X is smooth and $G = e$. Then applying π_0 at level zero in this diagram gives

$$\begin{array}{ccc} K_{\mathrm{alg}}(X)_o & \longrightarrow & \{1\} \times \prod_{k=n-1}^{0} \mathcal{A}_k(X) \\ \big\downarrow & & \big\downarrow \\ K(X)_o & \longrightarrow & H^{2*}(X; \mathbb{Z}). \end{array}$$

where the top horizontal map is the total Chern class map into algebraic cycles modulo algebraic equivalence under intersection product [5], and the bottom horizontal arrow is the standard total Chern class [6]. The preceding diagram of G-spectra is a full extension of this classical diagram to equivariant theories.

Appendix A. The Coherence Theorem

The point of this appendix is to provide the following.

Proof. [of Proposition 2.3.1]

We first observe that any morphism $\theta : H \to H'$ in $\mathcal{G}$ has one of the following three forms:

(1) $\theta = \alpha_{H',H} : H \hookrightarrow H'$ is an inclusion of subgroups;
(2) $\theta = \beta_H : H \to \{e\}$ is a projection onto the trivial subgroup;
(3) θ is a composition of the type $H \xrightarrow{\beta_H} \{e\} \xrightarrow{\alpha_{H',e}} H'$.

Therefore, we only need to define η_θ in the cases (1) and (2), satisfying suitable conditions, and use the natural equivalence of functors $\beta^\sharp \alpha'^\sharp \cong (\alpha'\beta)^\sharp = \theta^\sharp$ to define η_θ in the third case by the commutativity of the diagram

$$\begin{array}{ccc} \beta^\sharp {\alpha'}^\sharp \mathbf{k}_{H'} & \xrightarrow{\beta^\sharp \eta_{\alpha'}} & \beta^\sharp \mathbf{k}_e \\ \cong \Big\downarrow & & \Big\downarrow \eta_\beta \\ \theta^\sharp = (\alpha'\beta)^\sharp \mathbf{k}_{H'} & \xrightarrow[\eta_{\alpha'\beta} = \eta_\theta]{} & \mathbf{k}_H, \end{array}$$

where $\alpha' = \alpha_{H',e}$ and $\beta = \beta_H$.

Let us first recall how $\mathbf{k}_H$ is defined. Given an H-module $V \subset \mathcal{U}_H$, let D^H_V denote the monad associated to the operad $\mathcal{D}^H_V = \mathcal{C}_V \times \mathcal{L}(\mathcal{U}_H)$, where $\mathcal{C}_V$ is the little discs operad (cf. [4, 2.2(b)]). By definition

$$\mathbf{k}_H(V) = \lim_{\substack{W \subset \mathcal{U}_H \\ V \subset W}} \Omega^{W-V} B(\Sigma^V, D^H_V, X_H). \tag{A.1}$$

The first case ($\theta = \alpha_{H',H} : H \hookrightarrow H'$) is quite simple, since $j_\theta : \theta^* \mathcal{U}_{H'} \longrightarrow \mathcal{U}_H$ is the identity map regarded as an H-module map. We will define a map $\tau_\theta : \theta^* \mathbf{k}_{H'} \longrightarrow j^*_\theta \mathbf{k}_H$ in $H\mathcal{S}(\theta^*\mathcal{U}_{H'})$ and then apply the functor $j_{\theta *}$ to τ_θ to obtain $\eta_\theta : \theta^\sharp \mathbf{k}_{H'} = j_{\theta *} \theta^* \mathbf{k}_{H'} \xrightarrow{j_{\theta *}(\tau_\theta)} (j_{\theta *} j^*_\theta \mathbf{k}_H) = \mathbf{k}_H$. The last identification comes from the fact that $j_{\theta *} = (j_\theta^{-1})^*$ in this particular situation.

The map $\tau_\theta : \theta^* \mathbf{k}_{H'} \longrightarrow j^*_\theta \mathbf{k}_H$ is defined as follows. Given $V \subset \theta^* \mathcal{U}_{H'}$ then $(\theta^* \mathbf{k}_{H'})(V) = \mathbf{k}_{H'}(V)$ as an H-space. On the other hand, there is a natural identification of monads $D^H_V = D^{H'}_V$ in the category of H-spaces, when V is seen as an H-module and $\mathcal{L}(\mathcal{U}_{H'})$ is identified to $\mathcal{L}(\mathcal{U}_H)$ as an H-operad. Therefore the identification $\theta^* X_{H'} \longrightarrow X_H$ induces compatible homeomorphisms of H-spaces $\Omega^{W-V} B(\Sigma^W, D^{H'}_W, X_{H'}) \longrightarrow \Omega^{W-V} B(\Sigma^W, D^H_W, X_H)$ for any $W \subset \mathcal{U}_H$ such that $V \subset W$. Furthermore, once one observes that the family of H'-submodules of $\mathcal{U}_{H'}$ is cofinal in the family of H-submodules of $\mathcal{U}_{H'}$, then the above maps induce an isomorphism of spectra $\tau_\theta : \theta^* \mathbf{k}_{H'} \longrightarrow j_{\theta *} \mathbf{k}_H$ when one passes to the appropriate limit; cf. (A.1). Therefore, $\eta_\theta = j_{\theta *}(\tau_\theta)$ is also an isomorphism, in this case. It is a tautology to check that when $\theta : H \longrightarrow H'$ and $\theta' : H' \longrightarrow H''$ are inclusions, then the desired diagram commutes.

The definition of η_θ, when $\theta = \beta_H : H \longrightarrow \{e\}$, is a bit more involved. Given $W \subset \mathcal{U}_e$, denote by $\tilde{\mathcal{D}}^H_W$ the operad $\mathcal{C}_W \times \mathcal{L}(\mathcal{U}_H)^H$, and let $\tilde{D}^H_W$ be its associated monad. The inclusion $\tilde{\mathcal{D}}^H_W \longrightarrow \mathcal{D}^e_W$ is a local equivalence (see [18, §3]) of E_∞-operads which induces equivalences of spaces:

$$B(\Sigma^W, \tilde{D}^H_W, X_e) \xrightarrow{\cong} B(\Sigma^W, D^e_W, X_e),$$

for all $W \subset \mathcal{U}_e$. These equivalences assemble to define

$$\Phi_V : \lim_{\substack{W \subset \mathcal{U}_e \\ V \subset W}} \Omega^{W-V} B(\Sigma^W, \tilde{D}^H_W, X_e) \xrightarrow{\cong} \lim_{\substack{W \subset \mathcal{U}_e \\ V \subset W}} \Omega^{W-V} B(\Sigma^W, D^e_W, X_e) := \mathbf{k}_e(V). \tag{A.2}$$

We now introduce a series of intermediate natural maps. The given map $f_H : X_e \longrightarrow X^H_H$ of $\mathcal{L}(\mathcal{U}_H)^H$-spaces induces maps

$$B(\Sigma^W, \tilde{D}^H_W, X_e) \rightarrow B(\Sigma^W, \tilde{D}^H_W, X^H_H)$$

for each $W \subset \mathcal{U}_e$, which in turn define

$$f_{H*} : \lim_{\substack{W\subset\mathcal{U}_e \\ V\subset W}} \Omega^{W-V} B(\Sigma^W, \tilde{D}^H_W, X_e) \;\to\; \lim_{\substack{W\subset\mathcal{U}_e \\ V\subset W}} \Omega^{W-V} B(\Sigma^W, \tilde{D}^H_W, X^H_H). \tag{A.3}$$

On the other hand, the inclusions $\mathcal{L}(\mathcal{U}_H)^H \hookrightarrow \mathcal{L}(\mathcal{U}_H)$ and $j_\beta : \beta^*\mathcal{U}_e \to \mathcal{U}_H$ induce maps

$$B(\Sigma^W, \tilde{D}^H_W, X^H_H) \;\to\; B(\Sigma^{j_\beta W}, D^H_{j_\beta W}, X^H_H)^H,$$

for each $W \subset \mathcal{U}_e$, with compatible associated maps

$$\Omega^{W-V} B(\Sigma^W, \tilde{D}^H_W, X^H_H) \;\to\; \left\{\Omega^{j_\beta W - j_\beta V} B(\Sigma^{j_\beta W}, D^H_{j_\beta W}, X^H_H)\right\}^H.$$

One then obtains, by passing to limits on both sides, a map:

$$\Psi_V : \lim_{\substack{W\subset\mathcal{U}_e \\ V\subset W}} \Omega^{W-V} B(\Sigma^W, \tilde{D}^H_W, X^H_H) \to \lim_{\substack{U\subset\mathcal{U}_H \\ j_\beta(V)\subset U}} \left\{\Omega^{U - j_\beta V} B(\Sigma^U, D^H_U, X^H_H)\right\}^H. \tag{A.4}$$

Finally, one can use j_β to define a non-equivariant map

$$\Omega^{W-V} B(\Sigma^W, D^e_W, X_e) \;\to\; \Omega^{j_\beta W - j_\beta V} B(\Sigma^{j_\beta W}, D^H_{j_\beta W}, X_H),$$

where X_e is identified with X_H and D^e_W with $D^H_{j_\beta W}$ when the action of H is ignored. This induces the equivalence

$$\Xi_V : \mathbf{k}_e(V) = (\beta\alpha)_\sharp \mathbf{k}_e(V) \to \beta_\sharp\alpha_\sharp\mathbf{k}_e(V) = \lim_{\substack{U\subset\mathcal{U}_H \\ j_\beta(V)\subset U}} \Omega^{U - j_\beta V} B(\Sigma^U, D^H_U, X_H), \tag{A.5}$$

where we use the particular description of $j_\alpha = j_{\alpha_{H,e}}$ to obtain the equality on the right hand side of the equation. The following diagram commutes

$$\begin{tikzcd}
\lim_{\substack{W\subset\mathcal{U}_e \\ V\subset W}} \Omega^{W-V} B(\Sigma^W, \tilde{D}^H_W, X_e) \arrow[r, "\Phi_V", "(A.2)"'] \arrow[d, "f_{H*}"', "(A.3)"] & \lim_{\substack{W\subset\mathcal{U}_e \\ V\subset W}} \Omega^{W-V} B(\Sigma^W, D^e_W, X_e) \arrow[d, equal] \\
\lim_{\substack{W\subset\mathcal{U}_e \\ V\subset W}} \Omega^{W-V} B(\Sigma^W, \tilde{D}^H_W, X^H_H) \arrow[d, "\Psi_V"', "(A.4)"] & \mathbf{k}_e(V) \arrow[d, "(A.5)"', "\Xi_V"] \\
\lim_{\substack{U\subset\mathcal{U}_H \\ V\subset U}} \left\{\Omega^{U-j_\beta V} B(\Sigma^U, D^H_U, X_H)\right\}^H \arrow[r, "\subset"'] \arrow[d, equal] & \lim_{\substack{U\subset\mathcal{U}_H \\ j_\beta V\subset U}} \Omega^{U-j_\beta V} B(\Sigma^U, D^H_U, X_H) \arrow[d, equal] \\
\beta_\sharp \mathbf{k}_H(V) \arrow[r, "\beta_\sharp\xi_\alpha"] & \beta_\sharp\alpha_\sharp\mathbf{k}_e(V),
\end{tikzcd}$$

where $\xi_\alpha : \mathbf{k}_e \to \alpha_\sharp\mathbf{k}_H$ is the adjoint of η_α, with $\alpha = \alpha_{H,e}$.

Define $\xi_{\beta,V} := \Psi_V \circ f_{H*} \circ \Phi_V^{-1}$ and notice that we have proven that the diagram

$$\begin{tikzcd}
\mathbf{k}_e(V) \arrow[r, "id"] \arrow[d, "\xi_{\beta,V}"'] & (\beta\alpha)_\sharp\mathbf{k}_e(V) \arrow[d, "\cong"', "\Xi_V"] \\
\beta_\sharp\mathbf{k}_H(V) \arrow[r, "\beta_\sharp\xi_\alpha"'] & \beta_\sharp\alpha_\sharp\mathbf{k}_e(V)
\end{tikzcd}$$

commutes. If $\eta_\beta : \beta^\sharp \mathbf{k}_e \longrightarrow \mathbf{k}_H$ denotes the adjoint of the map of spectra defined by ξ_β, we conclude that the adjoint diagram

$$\begin{array}{ccc} \beta^\sharp(\alpha^\sharp \mathbf{k}_e) & \xrightarrow{\beta^\sharp \eta_\alpha} & \beta^\sharp \mathbf{k}_H \\ \cong \big\downarrow & & \big\downarrow \eta_\beta \\ (\alpha\beta)^\sharp \mathbf{k}_e & \xrightarrow{\eta_{\alpha\beta}} & \mathbf{k}_e \end{array}$$

also commutes. A simple verification now shows the commutativity of all required diagrams. □

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