

# ON THE COMPLEMENT OF THE PROJECTIVE HULL IN $\mathbf{C}^n$ .

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## ABSTRACT

We prove that if  $K$  is a compact subset of an affine variety  $\Omega = \mathbf{P}^n - D$  (where  $D$  is a projective hypersurface) and if  $K$  is contained in a closed analytic subvariety  $V \subset \Omega$ , then the projective hull  $\widehat{K}$  has the property that  $\widehat{K} \cap \Omega \subset V$ . If  $V$  is smooth and 1-dimensional, then  $\widehat{K} \cap \Omega$  is also closed in  $\Omega$ . The result has applications to graphs in  $\mathbf{C}^2$  of functions in the disk algebra.

Let  $X$  be a compact set in  $\mathbf{C}^n$ . In [1], R. Harvey and the first author introduce a generalization of the polynomial hull of  $X$ , which they call the *projective hull of  $X$  in  $\mathbf{C}^n$* , and denote by  $\widehat{X}$ . For  $d = 1, 2, \dots$  let  $\mathcal{P}_d$  denote the space of all polynomials on  $\mathbf{C}^n$  of degree  $\leq d$ . Then by definition a point  $z$  lies in  $\widehat{X}$  if there exists a constant  $C \geq 1$  such that for all  $d$  and for all  $P \in \mathcal{P}_d$  with  $\sup_X |P| \leq 1$ , we have

$$|P(z)| \leq C^d.$$

Fix a point  $a = (a_1, \dots, a_n) \in \mathbf{C}^n - X$ . For  $d = 1, 2, \dots$  set

$$\lambda_d(a) \equiv \sup\{|P(a)| : P \in \mathcal{P}_d \text{ with } \|P\|_X \leq 1\}$$

Note that  $a$  lies in the complement of  $\widehat{X}$  if and only if for each  $M \geq 1$  there exist arbitrarily large  $d$  with  $\lambda_d(a) \geq M^d$ .

We call a set *non-algebraic* if every polynomial vanishing on  $X$  is identically zero. We restrict our attention to such non-algebraic sets. We study the function  $\lambda_d$  and, in Theorem 2 below, give a result on the complement of  $\widehat{X}$  in certain cases, generalizing Theorem 9.2 in [1].

**THEOREM 1.** *Let  $X$  be as above. Fix  $M \geq 1$ . Fix  $d$  and let  $\lambda_d = \lambda_d(0)$ . Then  $\lambda_d \geq M^d$  if and only if there exist polynomials  $A_1, \dots, A_n$  on  $\mathbf{C}^n$  such that  $\zeta_1 A_1(\zeta) + \dots + \zeta_n A_n(\zeta) \in \mathcal{P}_d$  with  $|1 - \sum_k \zeta_k A_k| \leq \frac{1}{M^d}$  on  $X$ .*

We use the following Banach distance formula.

**LEMMA 1.** *Let  $W$  be a normed linear space,  $\dim(W) = N$ , and let  $\mathcal{V}$  be a subspace of  $W$  with  $\dim(\mathcal{V}) = N - 1$ . Fix  $x \in W - \mathcal{V}$ . Let  $L$  be the linear functional on  $W$  with  $L = 0$  on  $\mathcal{V}$  and  $L(x) = 1$ . Let  $\delta = \text{dist}(x, \mathcal{V})$ . Then*

$$\|L\| = \frac{1}{\delta}.$$

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\*Partially supported by the N.S.F.

NOTE . See the Appendix for a proof.

**Proof of Theorem 1.** Let  $\mathcal{W} = \mathcal{P}_d$ . For  $P \in \mathcal{P}_d$  put  $\|P\| = \sup_X |P|$ . Note that  $\|P\| = 0$  only when  $P = 0$  by the hypothesis that  $X$  is non-algebraic.

A basis for  $\mathcal{W}$  is the set of monomials  $\zeta^\alpha = \zeta_1^{\alpha_1} \zeta_2^{\alpha_2} \cdots \zeta_n^{\alpha_n}$  with  $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq d$ . Let  $\mathcal{V}$  be the subspace of  $\mathcal{W}$  spanned by  $\{\zeta^\alpha : |\alpha| > 0\}$ . Let  $x$  denote the constant polynomial 1. Define  $L$  to be the linear functional  $L : \mathcal{W} \rightarrow \mathbf{C}$  given by  $L(P) = P(0)$ . Then  $\lambda_d = \|L\|$  in the dual space of  $\mathcal{W}$ . Put  $\delta =$  the distance from 1 to  $\mathcal{V}$ , so

$$\delta = \inf \left\| 1 - \sum_{0 < |\alpha| \leq d} c_\alpha \zeta^\alpha \right\|.$$

By Lemma 1,  $\lambda_d \geq M^d$  if and only if  $\|L\| \geq M^d$  if and only if  $\frac{1}{\delta} \geq M^d$  if and only if  $\delta \leq \frac{1}{M^d}$  if and only if

$$\left| 1 - \sum_{k=1}^n \zeta_k A_k \right| \leq \frac{1}{M^d} \quad \text{on } X$$

for polynomials  $A_1, \dots, A_n \in \mathcal{P}_{d-1}$ . We are done. ■

**THEOREM 2.** *Let  $\Sigma$  be a closed complex analytic subvariety of  $\mathbf{C}^n$ . Let  $K$  be a compact set contained in  $\Sigma$ . Then  $\widehat{K} \subset \Sigma$ .*

NOTE 1. In the case that  $\Sigma \subset \mathbf{C}^2$  is the graph of an entire function  $f$  on  $\mathbf{C}$ , with  $f$  not a polynomial, the result that  $K \subset \Sigma$  implies  $\widehat{K} \subset \Sigma$  is proved in Theorem 9.2 in [1].

**Proof of Theorem 2.** Fix  $a = (a_1, \dots, a_n) \in \mathbf{C}^n - \Sigma$ . By definition  $\lambda_d(a) = \sup |P(a)|$  taken over all  $P \in \mathcal{P}_d$  with  $\|P\|_K \leq 1$ . We make the following

**Claim:** Fix  $M \geq 1$ . Then  $\lambda_d(a) > M^d$  for all large  $d$ .

**Proof of Claim:**

**Case 1:**  $a = 0$  so  $\lambda_d(a) = \lambda_d$ . Then  $0 \in \mathbf{C}^n - \Sigma$ . It is known that we may choose a function  $H_1$ , holomorphic on  $\mathbf{C}^n$  with  $H_1 = 0$  on  $\Sigma$  and  $H_1(0) = 1$ . (See, for example, [2 Ch. VIII.A, Thm. 18].) Put  $H = 1 - H_1$ . Then  $H$  is holomorphic on  $\mathbf{C}^n$  with  $H = 1$  on  $\Sigma$  and  $H(0) = 0$ . Now  $H$  has a power series expansion

$$H(\zeta) = \sum_{\alpha} c_\alpha \zeta^\alpha$$

convergent on  $\mathbf{C}^n$ . Fix  $R > 0$ . There exists a constant  $C_R$  such that

$$|c_\alpha| \leq \frac{C_R}{R^{|\alpha|}} \quad \text{for all } \alpha$$

Fix  $d$ . Put  $P_d(\zeta) = \sum_{|\alpha| \leq d} c_\alpha \zeta^\alpha$  and put  $\mathcal{E}_d(\zeta) = \sum_{|\alpha| > d} c_\alpha \zeta^\alpha$ . Note that  $P_d(0) = 0$ . Then  $H = P_d + \mathcal{E}_d$  on  $\mathbf{C}^n$ . In particular,  $1 = P_d + \mathcal{E}_d$  on  $K$ , so

$$|1 - P_d| \leq |\mathcal{E}_d| \quad \text{on } K. \tag{1}$$

Fix  $R_0$  so that  $K \subset \{\zeta : |\zeta_k| \leq R_0 \text{ for } 1 \leq k \leq n\}$ . Choose  $\zeta \in K$ . Then

$$|\mathcal{E}_d(\zeta)| \leq \sum_{|\alpha|>d} |c_\alpha| |\zeta^\alpha| \leq \sum_{|\alpha|>d} |c_\alpha| R_0^{|\alpha|}.$$

Fix  $R$  such that  $\frac{R_0}{R} \leq \frac{1}{2M}$ . Since  $|c_\alpha| \leq \frac{C_R}{R^{|\alpha|}}$ , we have

$$\sum_{|\alpha|>d} |c_\alpha| R_0^{|\alpha|} \leq C_R \sum_{|\alpha|>d} \left(\frac{R_0}{R}\right)^{|\alpha|}$$

Put  $t = \frac{R_0}{R}$ . Then

$$\sum_{|\alpha|>d} t^{|\alpha|} = \sum_{k=d+1}^{\infty} \sum_{|\alpha|=k} t^{|\alpha|} = \sum_{k=d+1}^{\infty} \binom{n+k-1}{n-1} t^k \leq \frac{1}{M^d} \quad \text{for large } d.$$

Hence,

$$|\mathcal{E}_d(\zeta)| \leq \frac{C_R}{M^d} \leq \frac{1}{(M/2)^d} \quad \text{for large } d. \quad (2)$$

From (1) and Theorem 1 we now get

$$\lambda_d \geq \left(\frac{M}{2}\right)^d \quad \text{for large } d.$$

Since  $M$  was arbitrary, the claim is proved Case 1.

**Case 2:**  $a \in \mathbf{C}^n - \Sigma$ . Let  $\chi : \mathbf{C}^n \rightarrow \mathbf{C}^n$  be the translation  $\chi(\zeta) = \zeta - a$ . Then  $\chi(a) = 0$  and  $\chi$  takes the space  $\mathcal{P}_d$  isomorphically onto itself. Put  $K' = \chi(K)$  and  $\Sigma' = \chi(\Sigma)$ . Since  $a \notin \Sigma$ ,  $0 \notin \Sigma'$  and so by Case 1 there exists  $P' \in \mathcal{P}_d$  with  $\|P'\|_{K'} \leq 1$ ,  $|1 - P'| \leq \frac{1}{M^d}$  on  $K'$  and  $P'(0) = 0$ .

By Case 1,  $\lambda_d \geq M^d$  for large  $d$ , so there exists a polynomial  $Q'$  in  $\mathcal{P}_d$  with  $\|Q'\|_{K'} \leq 1$  and  $|Q'(0)| \geq M^d$ . Now put  $Q = Q' \circ \chi$ . Then  $Q \in \mathcal{P}_d$ ,  $|Q(a)| \geq M^d$  and  $\|Q\|_K \leq 1$ . so  $\lambda_d(a) \geq M^d$ . Case 2 is done and so the claim holds.

If  $a \in \mathbf{C}^n - \Sigma$ , then  $a$  does not belong to  $\widehat{K}$ . So  $\widehat{K} \subset \Sigma$ . We are done. ■

**COROLLARY 1.** *Let  $V$  be a closed complex submanifold of dimension one in  $\mathbf{C}^n$  and  $K \subset V$  a compact subset. Then  $\widehat{K}$  is a closed subset of  $\mathbf{C}^n$ .*

**NOTE.** It is not true in general that  $K$  compact in  $\mathbf{C}^n$  implies that  $\widehat{K}$  is closed in  $\mathbf{C}^n$ .

**Proof of Corollary 1.** In the notation of [1], if  $K$  is a compact set in  $\mathbf{C}^n$ ,  $\mathcal{S}_K$  is the family of functions  $\varphi = \frac{1}{d} \log|P|$ ,  $P \in \mathcal{P}_d$ ,  $|P| \leq 1$  on  $K$ . We define the extremal function

$$\Lambda_K(x) = \sup_{\varphi \in \mathcal{S}_K} \varphi(x).$$

Then for  $x \in \mathbf{C}^n$ ,  $\Lambda_K(x) < \infty \iff x \in \widehat{K}$ . That is,  $\widehat{K} \cap \mathbf{C}^n = \{x \in \mathbf{C}^n : \Lambda_K(x) < \infty\}$ . Since  $\Lambda_K \equiv \infty$  in  $\mathbf{C}^n - V$ , we conclude from Theorem 7.3 in [1] that in every connected component of  $V_K$  either  $\Lambda_K \equiv \infty$  or  $\Lambda_K$  is a locally bounded harmonic function.

Suppose now that  $\{x_\nu\}$  is a sequence of points in  $\widehat{K}$  converging to a point  $x$ . Then  $x \in V$ . We must show that  $x \in \widehat{K}$ .

We may assume that  $x \notin K$ , so  $x$  lies in some connected component  $\Omega$  of  $V - K$ . For large  $\nu$ ,  $x_\nu \in \Omega$ . Since  $\Lambda_K(x_\nu) < \infty$ , we have by Theorem 7.3 of [1] quoted above, that  $\Lambda_K$  is locally bounded on  $\Omega$ . In particular,  $L_K(x)$  is finite, and so  $x \in \widehat{K}$ . ■

NOTE 2. The projective hull, as introduced in [1], is a subset  $\widehat{K} \subset \mathbf{P}^n$  associated to any compact set  $K \subset \mathbf{P}^n$ . In an affine chart  $\mathbf{C}^n = \mathbf{P}^n - \mathbf{P}^{n-1}$ , the set  $\widehat{K} \cap \mathbf{C}^n$  agrees with the definition given above. In what follows  $\widehat{K}$  will refer to this full projective hull.

COROLLARY 2. *Let  $D \subset \mathbf{P}^n$  be a complex hypersurface in complex projective space, and suppose  $\Sigma \subset \mathbf{P}^n - D$  is a closed subvariety of the complement. If  $K$  is a compact subset of  $\Sigma$ , then  $\widehat{K} \cap (\mathbf{P}^n - D) \subset \Sigma$ . Moreover, if  $\Sigma$  is smooth of dimension one, then  $\widehat{K} \cap (\mathbf{P}^n - D)$  is closed in  $\mathbf{P}^n - D$ .*

**Proof.** The divisor  $D$  can be realized as a hyperplane section  $D = v(\mathbf{P}^n) \cap \mathbf{P}^{N-1}$  under a Veronese embedding  $v : \mathbf{P}^n \rightarrow \mathbf{P}^N$ . By Proposition 3.2 in [1] we know that  $v(\widehat{K}) = \widehat{v(K)}$ . We now apply Theorem 2 to  $v(K) \subset v(\Sigma) \subset \mathbf{C}^N = \mathbf{P}^N - \mathbf{P}^{N-1}$ , and the first statement follows. If  $\Sigma$  is a smooth curve, then  $v(\widehat{K})$  is closed in  $\mathbf{C}^N$  by Corollary 1. Hence,  $v^{-1}(v(\widehat{K})) = v^{-1}(v(\widehat{K})) = \widehat{K}$  is closed in  $\mathbf{P}^n - D$ . ■

COROLLARY 3. *Let  $K \subset \Sigma \subset \mathbf{P}^n - D$  be as in Corollary 2. Suppose  $\widehat{K} \cap (\mathbf{P}^n - D) \subset \subset \mathbf{P}^n - D$ . Then  $\widehat{K} \cap D = \emptyset$ .*

**Proof.** Suppose there exists a point  $x \in \widehat{K} \cap D$ . Then by Theorem 11.1 in [1], there exists a positive current  $T$  of bidimension (1,1) with

$$\text{supp}T \subset \widehat{K}^- \quad \text{and} \quad dd^c T = \mu - \delta_x \quad (3)$$

where  $\mu$  is a probability measure on  $K$  and  $\widehat{K}^-$  denotes the closure of  $K$ . Let  $U, V \subset \mathbf{P}^n$  be disjoint open subsets with  $\widehat{K} \cap (\mathbf{P}^n - D) \subset U$  and  $D \subset V$ . Then by our hypothesis and the first part of (3), we have  $T = \chi_U T + \chi_V T$ . We conclude from the second part of (3) that  $dd^c(\chi_U T) = \mu$  and  $dd^c(\chi_V T) = -\delta_x$ , both of which are impossible since  $(dS, 1) = 0$  for any current  $S$  of dimension  $2n - 1$ . ■

COROLLARY 4. *Suppose  $K = \gamma$  is a disjoint union of real closed curves contained in a smooth complex analytic (non-algebraic) curve  $\Sigma \subset \mathbf{P}^n - D$  with  $D$  as above. Suppose that  $\Sigma - \gamma$  has only one unbounded component. Then  $\widehat{\gamma} - \gamma$  is exactly the union of the bounded components of  $\Sigma - \gamma$ .*

**Proof.** By Corollary 2,  $\widehat{\gamma} \cap (\mathbf{P}^n - D) \subset \Sigma$ . As seen in the proof of Corollary 1, if one point  $x \in \Sigma - \gamma$  belongs to  $\widehat{\gamma}$ , then the entire connected component of  $x$  in  $\Sigma - \gamma$  belongs to  $\widehat{\gamma}$ . By [1, Cor. 3.4] every bounded component of  $\Sigma - \gamma$  lies in  $\widehat{\gamma}$ . By Sadullaev's Theorem (cf. [1, §7]), if the unbounded component of  $\Sigma - \gamma$  lies in  $\widehat{\gamma}$ , then  $\Sigma$  is algebraic. Thus  $\widehat{\gamma} \cap (\mathbf{P}^n - D)$  is exactly the union of the bounded components of  $\Sigma - \gamma$ . We now apply Corollary 3. ■

NOTE 3. We can find applications of Corollary 4 by considering graphs of certain functions in the disk algebra.

Let  $\Gamma$  be the unit circle in the  $\zeta$ -plane and let  $A(\Gamma) = \{\varphi|_{\Gamma} : \varphi \text{ is analytic in } |\zeta| < 1 \text{ and continuous in } |\zeta| \leq 1\}$ . Let  $K$  denote the graph of  $\varphi$  over  $\Gamma$ , so

$$K = \{(\zeta, \varphi(\zeta)) : \zeta \in \Gamma\} \subset \mathbf{C}^2.$$

In the following examples the projective hull of  $K$  in  $\mathbf{P}^2$  is  $\widehat{K} = \{(\zeta, \varphi(\zeta)) : |\zeta| \leq 1\}$ .

- (1) Let  $\varphi(\zeta) = \log(\zeta - 2)$ . Then  $K = \{(\zeta, \log(\zeta - 2)) : \zeta \in \Gamma\}$ , so  $K$  is contained in the closed complex submanifold  $\Sigma = \{(\zeta, w) : e^w = \zeta - 2\}$  of  $\mathbf{C}^2$ .
- (2) Let  $\varphi(\zeta) = \sqrt{e^{2\zeta} + 5\zeta + 20}$  and  $\Sigma = \{(\zeta, w) : w^2 = e^{2\zeta} + 5\zeta + 20\}$ . The graph  $K$  of  $\varphi$  over  $\Gamma$  is contained in  $\Sigma$ .
- (3) Let  $\varphi(\zeta)$  be the restriction to  $\Gamma$  of a meromorphic function  $F$  defined on all of  $\mathbf{C}$  whose poles lies outside the unit disk. Write  $F = E_1/E_2$  where  $E_1$  and  $E_2$  are entire functions without common zeros. Then  $\Sigma = \{(\zeta, w) : E_2(\zeta)w = E_1(\zeta)\}$  (the graph of  $F$ ) is a closed submanifold of  $\mathbf{C}^2$  which contains  $K$ .
- (4) Let

$$\varphi(\zeta) = \sum_{\nu=1}^{\infty} \frac{c_{\nu}}{\zeta - a_{\nu}}$$

where  $\sum_{\nu} |c_{\nu}| < \infty$ , each  $|a_{\nu}| > 1$ , and  $\lim_{\nu \rightarrow \infty} a_{\nu} = a$  with  $|a| > 1$ . Then  $K$  is contained in  $\Sigma = \{(\zeta, \varphi(\zeta)) : \zeta \neq a \text{ or } a_{\nu} \text{ for any } \nu\}$ , which is a closed submanifold of  $\mathbf{C}^2 - \{\zeta = a\} = \mathbf{P}^2 - \mathbf{P}_{\infty}^1 - \{\zeta = a\}$ .

### Appendix: Proof of the Banach Distance Formula.

Fix  $f \in \mathcal{V}$ . Then  $L(x - f) = L(x) = 1$ . Hence,  $1 = |L(x - f)| \leq \|L\| \cdot \|x - f\|$ . Hence,  $1 \leq \|L\| \cdot \delta$ , so  $\|L\| \geq \frac{1}{\delta}$ .

Next, fix  $g \in \mathcal{W}$ ,  $g = f + tx$  for  $t \in \mathbf{C} - \{0\}$ ,  $f \in \mathcal{V}$ . We have  $L(g) = t$ . Now  $g = t(x + \frac{f}{t})$ , so  $\|g\| = |t| \|x + \frac{f}{t}\| \geq |t| \cdot \delta$ , since  $-\frac{f}{t} \in \mathcal{V}$ . So  $|L(g)| = |t| \leq \frac{1}{\delta} \|g\|$ . Thus,  $\|L\| \leq \frac{1}{\delta}$ , and therefore  $\|L\| = \frac{1}{\delta}$ . ■

### References.

- [1] F. R. Harvey and H. B. Lawson, Jr. *Projective hulls and the projective Gelfand transform*, Asian J. Math. **10** (2006), 607-646.
- [2] R. C. Gunning and H. Rossi, *Analytic Functions of Several Complex Variables*, Prentice-Hall, Englewood Cliffs, N. J., 1965.