Jim Simons Work on the Theory of Minimal Varieties
Jim Simons Work
on the
Theory of Minimal Varieties
In 1967

Jim Simons wrote a remarkable paper

on the subject of

Minimal Submanifolds in Riemannian Geometry
In 1967

Jim Simons wrote a remarkable paper

on the subject of

Minimal Submanifolds in Riemannian Geometry

Minimal varieties in riemannian manifolds

By JAMES SIMONS

CONTENTS

§1. Introduction

§2. Riemannian vector bundles
  1.1 Definitions
  1.2 Levi-Civita connection

§3. Immersed submanifolds
  3.1 Connections in the tangent and normal bundles
  3.2 The second fundamental form
  3.3 Curvature in the tangent and normal bundles

§4. Variations

§5. Minimal varieties
  5.1 Definitions and examples
  5.2 The first and second variation of area
  5.3 Jacobi fields
  5.4 The Morse index theorem
  5.5 Jacobi fields on Kähler submanifolds

§6. An estimate of the言行 between

§7. The fundamental elliptic equation
  7.1 The first order system
  7.2 The second order system
  7.3 The second fundamental form of minimal submanifolds

§8. Minimal varieties in spheres
  8.1 The index and the stability of a closed minimal variety
  8.2 An extrinsic rigidity theorem
  8.3 The fundamental equation and an intrinsic rigidity theorem
  8.4 Hessian shape differentials

§9. Minimal varieties in euclidean space
  9.1 Convex minimal varieties
  9.2 Plateau’s problem and the Bernstein conjecture

§10. Introduction

Our object in this paper is twofold. First, we give a basic exposition of immersed minimal varieties in a riemannian manifold. The principal result of this general investigation is the derivation of the linear elliptic second order equation satisfied by the second fundamental form of any minimal variety in any ambient manifold (cf. Theorem 4.2.1).

Second, we apply those general results in a more detailed study of minimal varieties in the sphere and in euclidean space. This study includes an estimation of a lower bound for the index and the nullity of a non-totally geodesic closed minimal variety immersed in $E^n$; a theorem which generalizes to arbitrary codimensions the theorem of De Giorgi [8] concerning the image
In this paper, he

- Established the foundations of the subject.
- Derived the fundamental elliptic system of PDE's governing the second fundamental form.
- Established complete interior regularity for minimizing hypersurfaces in dimensions $\leq 7$.
- Established the Bernstein Conjecture in dimensions $\leq 8$.
- Produced the example which eventually showed that both of the above theorems were sharp.
In this paper, he

- Established the foundations of the subject.
- Derived the fundamental elliptic system of pde’s governing the second fundamental form.
In this paper, he

- Established the foundations of the subject.
- Derived the fundamental elliptic system of pde’s governing the second fundamental form.
- Established complete interior regularity for minimizing hypersurfaces in dimensions $\leq 7$. 

Blaine Lawson
Jim Simons’ Work on Minimal Varieties
May 24, 2013
In this paper, he

- Established the foundations of the subject.
- Derived the fundamental elliptic system of pde's governing the second fundamental form.
- Established complete interior regularity for minimizing hypersurfaces in dimensions $\leq 7$.
- Established the Bernstein Conjecture in dimensions $\leq 8$. 

Blaine Lawson
Jim Simons' Work on Minimal Varieties
In this paper, he

- Established the foundations of the subject.
- Derived the fundamental elliptic system of pde's governing the second fundamental form.
- Established complete interior regularity for minimizing hypersurfaces in dimensions $\leq 7$.
- Established the Bernstein Conjecture in dimensions $\leq 8$.
- Produced the example which eventually showed that both of the above theorems were sharp.
The Classical Theory of Minimal Surfaces

Riemann, Weierstrauss

Jim Simons' Work on Minimal Varieties
The Classical Theory of Minimal Surfaces

Riemann, Weierstraus
The Classical Theory of Minimal Surfaces

Riemann, Weierstrass

The Idea: Consider a smooth surface in Euclidean 3-space $\Sigma \subset \mathbb{E}^3$. 
The Classical Theory of Minimal Surfaces

Riemann, Weierstrass

The Idea: Consider a smooth surface in Euclidean 3-space $\Sigma \subset \mathbb{E}^3$.

Suppose that for every deformation $\Sigma_t$, ($\Sigma_0 = \Sigma$) in the interior, the area satisfies

$$A(\Sigma_t) \geq A(\Sigma).$$
The Classical Theory of Minimal Surfaces

Then for all such deformations:

\[ \frac{d}{dt} A(\Sigma_t) \bigg|_{t=0} = 0 \]
Then for all such deformations:

$$\frac{d}{dt} A(\Sigma_t) \bigg|_{t=0} = 0 \quad \text{and} \quad \frac{d^2}{dt^2} A(\Sigma_t) \bigg|_{t=0} \geq 0$$
If for all such deformations:

$$\left. \frac{d}{dt} A(\Sigma_t) \right|_{t=0} = 0$$

is called a Minimal Surface.

If in addition

$$\left. \frac{d^2}{dt^2} A(\Sigma_t) \right|_{t=0} \geq 0$$

is called a Stable Minimal Surface.
The Classical Theory of Minimal Surfaces

Riemann, Weierstrauss

If for all such deformations:

\[
\frac{d}{dt} A(\Sigma_t) \bigg|_{t=0} = 0
\]

\(\Sigma\) is called a **Minimal Surface**.
If for all such deformations:

\[ \frac{d}{dt} A(\Sigma_t) \bigg|_{t=0} = 0 \]

\( \Sigma \) is called a **Minimal Surface**.

If in addition

\[ \frac{d^2}{dt^2} A(\Sigma_t) \bigg|_{t=0} \geq 0 \]
If for all such deformations:

\[
\left. \frac{d}{dt} A(\Sigma_t) \right|_{t=0} = 0
\]

\(\Sigma\) is called a **Minimal Surface**.

If in addition

\[
\left. \frac{d^2}{dt^2} A(\Sigma_t) \right|_{t=0} \geq 0
\]

\(\Sigma\) is called a **Stable Minimal Surface**.
Observation:

These conditions

\[
\left. \frac{d}{dt} A(\Sigma_t) \right|_{t=0} = 0 \quad \text{and} \quad \left. \frac{d^2}{dt^2} A(\Sigma_t) \right|_{t=0} \geq 0
\]
Observation:

These conditions

\[ \left. \frac{d}{dt} A(\Sigma_t) \right|_{t=0} = 0 \quad \text{and} \quad \left. \frac{d^2}{dt^2} A(\Sigma_t) \right|_{t=0} \geq 0 \]

Make sense

- in arbitrary dimensions and codimensions,
Observation:

These conditions

\[ \frac{d}{dt} A(\Sigma_t) \bigg|_{t=0} = 0 \quad \text{and} \quad \frac{d^2}{dt^2} A(\Sigma_t) \bigg|_{t=0} \geq 0 \]

Make sense

- in arbitrary dimensions and codimensions,
- in general riemannian manifolds,
Observation:

These conditions

\[ \frac{d}{dt} A(\Sigma_t) \bigg|_{t=0} = 0 \quad \text{and} \quad \frac{d^2}{dt^2} A(\Sigma_t) \bigg|_{t=0} \geq 0 \]

Make sense

- in arbitrary dimensions and codimensions,
- in general riemannian manifolds,
- and for quite general objects \( \Sigma \)
A Geometric Characterization in $\mathbb{R}^3$

**THE GAUSS MAP**

$$N : \Sigma \rightarrow S^2$$

The Gauss map associates to each point $x \in \Sigma$, the normal vector $N(x)$ to $\Sigma$ at $x$, i.e., the vector perpendicular to the tangent plane to $\Sigma$ at $x$. 

Blaine Lawson

Jim Simons’ Work on Minimal Varieties

May 24, 2013 11 / 2
$\Sigma$ is **minimal** if and only if the Gauss map is (anti)-**conformal**.

Angles are preserved (but direction is reversed).
Complex Analysis Enters the Picture

Complex Analysis is a rich and deep subject with many beautiful results. We will return to this later.
Complex Analysis Enters the Picture

Take Stereographic Projection

Complex Analysis is a rich and deep subject with many beautiful results. We will return to this later.
Complex Analysis Enters the Picture

Take Stereographic Projection

Complex Analysis is a rich and deep subject with many beautiful results.
Suppose our surface \( \Sigma \) is the graph of a function \( z = f(x, y) \) over a domain \( D \) in the \((x, y)\)-plane.

When is this graph a minimal surface?
Elementary Question

Suppose our surface $\Sigma$ is the graph of a function $z = f(x, y)$ over a domain $D$ in the $(x, y)$-plane.

When is this graph a minimal surface?

ANSWER: It must satisfy the differential equation

\[(1 + f_y^2)f_{xx} + (1 + f_x^2)f_{yy} - 2f_x f_y f_{xy} = 0.\]
Suppose our surface $\Sigma$ is the graph of a function $z = f(x, y)$ over a domain $D$ in the $(x, y)$-plane.

When is this graph a minimal surface?

**ANSWER:** It must satisfy the differential equation

$$(1 + |\nabla f|^2)\Delta f - (\nabla f)^t \mathbf{H}(f) \nabla f = 0.$$
Let $D$ be the round disk of radius $R$. Let $\varphi$ be an arbitrary continuous function on the boundary circle.

**Theorem.** There exists a unique function $f(x, y)$ continuous on $D$ and smooth in its interior, such that $f = \varphi$ on $\partial D$ and in the interior it satisfies the minimal surface equation:

$$(1 + |\nabla f|^2) \Delta f - (\nabla f)^t H(f) \nabla f = 0$$
The Bernstein Theorem

This gives us a wildly abundant family of minimal surfaces which are graphs over disks of radius $R$. 

Surprise!!

The Bernstein Theorem (1918).

Any solution of the minimal surface equation which is defined for all $(x, y)$ in the plane must be linear, i.e., its graph is an affine 2-plane.
The Bernstein Theorem

This gives us a wildly abundant family of minimal surfaces which are graphs over disks of radius $R$.

Now imagine letting $R \to \infty$ and making clever choices for boundary curves $\varphi$.

Surprise!!
The Bernstein Theorem

This gives us a wildly abundant family of minimal surfaces which are graphs over disks of radius $R$.

Now imagine letting $R \to \infty$ and making clever choices for boundary curves $\varphi$.

One would expect to produce many functions $f(x, y)$ defined over the entire $(x, y)$-plane and satisfying the M.S.Eqn.
The Bernstein Theorem

This gives us a wildly abundant family of minimal surfaces which are graphs over disks of radius $R$.

Now imagine letting $R \to \infty$ and making clever choices for boundary curves $\varphi$.

One would expect to produce many functions $f(x, y)$ defined over the entire $(x, y)$-plane and satisfying the M.S.Eqn.

Surprise!!

The Bernstein Theorem (1918). Any solution of the minimal surface equation which is defined for all $(x, y)$ in the plane must be is linear, i.e., its graph is an affine 2-plane.
This is a beautiful and astonishing result.

If we remove a tiny disk from the plane, there is a function defined everywhere outside that disk whose graph is a minimal surface.
This is a beautiful and astonishing result.

If we remove a tiny disk from the plane, there is a function defined everywhere outside that disk whose graph is a minimal surface.

This is also true if we remove a half-line from the plane,
For the Classical Proof of this Theorem we return to the Gauss Map.

Notice: If $\Sigma = \{(x, y, f(x, y)) : (x, y) \in D\}$ is a graph of a function, then the image of the Gauss map lies in the upper hemisphere.
For the Classical Proof of this Theorem
we return to the Gauss Map.

Notice: If

\[ \Sigma = \{(x, y, f(x, y)) : (x, y) \in D\} \]

is a the graph of a function,
For the Classical Proof of this Theorem we return to the Gauss Map.

**Notice:** If

\[ \Sigma = \{(x, y, f(x, y)) : (x, y) \in D\} \]

is a the graph of a function, then

the image of the Gauss map lies in the upper hemisphere.
For the Classical Proof of this Theorem we return to the Gauss Map.

**Notice:** If

\[ \Sigma = \{(x, y, f(x, y)) : (x, y) \in D\} \]

is a the graph of a function, then

the image of the Gauss map lies in the upper hemisphere.

The proof is given by showing that \( \Sigma \) must have the conformal type of \( \mathbb{C} \).
For the Classical Proof of this Theorem we return to the Gauss Map.

**Notice:** If

$$\Sigma = \{(x, y, f(x, y)) : (x, y) \in D\}$$

is a the graph of a function, then

the image of the Gauss map lies in the upper hemisphere

The proof is given by showing that $\Sigma$ must have the conformal type of $\mathbb{C}$. Then the Gauss map becomes a bounded entire function and must be constant.
Question: Does this Theorem Generalize to Higher Dimensions?
Question:
Does this Theorem Generalize to Higher Dimensions?

FACT:
The graph

$$\Sigma = \{(x, f(z)) : z \in \mathbb{C}\} \subset \mathbb{C}^2$$

of any entire holomorphic function (e.g. a complex polynomial)
Question:
Does this Theorem Generalize to Higher Dimensions?

FACT:
The graph

\[ \Sigma = \{(x, f(z)) : z \in \mathbb{C}\} \subset \mathbb{C}^2 \]

of any entire holomorphic function (e.g. a complex polynomial)
is a (stable) minimal surface in \( \mathbb{C}^2 = \mathbb{R}^4 \).
Question:
Does this Theorem Generalize to Higher Dimensions?

FACT:

The graph

\[ \Sigma = \{(x, f(z)) : z \in \mathbb{C}\} \subset \mathbb{C}^2 \]

of any entire holomorphic function (e.g. a complex polynomial)

is a (stable) minimal surface in \( \mathbb{C}^2 = \mathbb{R}^4 \).

Restrict to codimension one.
The Bernstein Conjecture.

Let $f: \mathbb{R}^n \to \mathbb{R}$ be a solution of the minimal surface equation
\[ (1 + |\nabla f|^2) \Delta f - (\nabla f) \cdot H(f) \nabla f = 0 \]
defined over the entire space $\mathbb{R}^n$.

Then $f$ must be linear.
The Bernstein Conjecture.

Let

\[ f : \mathbb{R}^n \rightarrow \mathbb{R} \]

be a solution of the minimal surface equation

\[ (1 + |\nabla f|^2)\Delta f - (\nabla f)^t H(f) \nabla f = 0. \]

defined over the entire space \( \mathbb{R}^n \).
The Bernstein Conjecture.

Let

\[ f : \mathbb{R}^n \rightarrow \mathbb{R} \]

be a solution of the minimal surface equation

\[(1 + |\nabla f|^2) \Delta f - (\nabla f)^t H(f) \nabla f = 0.\]

defined over the entire space \( \mathbb{R}^n \).

Then \( f \) must be linear.
The Plateau Problem – in $\mathbb{R}^n$. 

Let $B \subset \mathbb{R}^n$ be a compact submanifold of dimension $p-1$ (without boundary).
The Plateau Problem – in $\mathbb{R}^n$.

Let

$$B \subset \mathbb{R}^n$$

be a compact submanifold of dimension $p - 1$ (without boundary).
The Plateau Problem – in $\mathbb{R}^n$.

Let

$$B \subset \mathbb{R}^n$$

be a compact submanifold of dimension $p - 1$ (without boundary).
The Plateau Problem – in $\mathbb{R}^n$.

**Problem:** Find a $p$-dimensional “submanifold” $\Sigma$ with “boundary” $B$
The Plateau Problem – in $\mathbb{R}^n$.

**Problem:** Find a $p$-dimensional “submanifold” $\Sigma$ with “boundary” $B$ such that

$$\mathcal{H}^p(\Sigma) \leq \mathcal{H}^p(\Sigma')$$

for all such $\Sigma'$ with boundary $B$. 
The Plateau Problem.

ISSUES:

• What do we mean by “submanifold”?
• What do we mean by “boundary”?
• Do solutions exist?
• How regular are the solutions?
The Plateau Problem.

ISSUES:

- What do we mean by “submanifold”? 
The Plateau Problem.

ISSUES:

- What do we mean by “submanifold”? 
- What do we mean by “boundary”?
The Plateau Problem.

ISSUES:

• What do we mean by “submanifold”?

• What do we mean by “boundary”?

• Do solutions exist?
The Plateau Problem.

ISSUES:

• What do we mean by “submanifold”?
• What do we mean by “boundary”?
• Do solutions exist?
• How regular are the solutions?
Consider a simple closed curve $\Gamma \subset \mathbb{R}^n$ and try to minimize area among all continuous maps of the disk $\psi: \Delta \to \mathbb{R}^n$ having first derivatives in $L^2$ and mapping $\psi: \partial \Delta \to \Gamma$ (monotonically).
The Plateau Problem – Classical Results.

Douglas and Rado 1930
Considered a simple closed curve $\Gamma \subset \mathbb{R}^n$.
The Plateau Problem – Classical Results.

Douglas and Rado 1930

Considered a simple closed curve $\Gamma \subset \mathbb{R}^n$

and tried to minimize area among all continuous maps of the disk

$$\psi : \Delta \rightarrow \mathbb{R}^n$$
The Plateau Problem – Classical Results.

Douglas and Rado 1930

Considered a simple closed curve $\Gamma \subset \mathbb{R}^n$

and tried to minimize area among all continuous maps of the disk $

\psi : \Delta \rightarrow \mathbb{R}^n$

having first derivatives in $L^2$ and mapping

$\psi : \partial \Delta \rightarrow \Gamma \quad \text{(monotonically)}$
The Plateau Problem – Classical Results.

Douglas and Rado 1930

However, self-intersections and isolated branch points could exist in these surfaces. (Osserman later proved that in $\mathbb{R}^3$ branch points do not exist.) Much work ensued – Courant, Morrey, etc. Surfaces of higher genus, with many boundary components in general manifolds.
The Plateau Problem – Classical Results.

Douglas and Rado 1930

established the existence of minimizers

(Osserman later proved that in $\mathbb{R}^3$ branch points do not exist.)

Much work ensued – Courant, Morrey, etc.

Surfaces of higher genus, with many boundary components in general manifolds.
The Plateau Problem – Classical Results.

Douglas and Rado 1930

established the existence of minimizers and the regularity of these minimizing maps (as mappings!)

(Osserman later proved that in $\mathbb{R}^3$ branch points do not exist.)

Much work ensued – Courant, Morrey, etc. Surfaces of higher genus, with many boundary components in general manifolds.
The Plateau Problem – Classical Results.

**Douglas and Rado 1930**

established the existence of minimizers and the regularity of these minimizing maps (as mappings!)

However, self-intersections and isolated branch points could exist in these surfaces.
The Plateau Problem – Classical Results.

Douglas and Rado 1930

established the existence of minimizers
and the regularity of these minimizing maps (as mappings!)
However, self-intersections and isolated branch points could exist in these surfaces.
(Osserman later proved that in $\mathbb{R}^3$ branch points do not exist.)
The Plateau Problem – Classical Results.

Douglas and Rado 1930

established the existence of minimizers and the regularity of these minimizing maps (as mappings!)

However, self-intersections and isolated branch points could exist in these surfaces.

(Osserman later proved that in $\mathbb{R}^3$ branch points do not exist.)

Much work ensued – Courant, Morrey, etc.

Surfaces of higher genus,
The Plateau Problem – Classical Results.

Douglas and Rado 1930

established the existence of minimizers
and the regularity of these minimizing maps (as mappings!)

However, self-intersections and isolated branch points could exist in these surfaces.

(Osserman later proved that in $\mathbb{R}^3$ branch points do not exist.)

Much work ensued – Courant, Morrey, etc.

Surfaces of higher genus,
with many boundary components
The Plateau Problem – Classical Results.

**Douglas and Rado 1930**

established the existence of minimizers
and the regularity of these minimizing maps (as mappings!)
However, self-intersections and isolated branch points could exist in these surfaces.
(Osserman later proved that in $\mathbb{R}^3$ branch points do not exist.)

**Much work ensued – Courant, Morrey, etc.**

Surfaces of higher genus,
with many boundary components
in general manifolds.
Reifenberg 1960
Reifenberg 1960

took a radically different approach to the Plateau Problem

Reifenberg 1960

took a radically different approach to the Plateau Problem
and proved the existence of solutions
among surfaces of all topological types.
Reifenberg 1960

took a radically different approach to the Plateau Problem
and proved the existence of solutions
among surfaces of all topological types.

He considered the family of all compact sets $\Sigma \subset \mathbb{R}^n$
with $B \subset \Sigma$
Reifenberg 1960

took a radically different approach to the Plateau Problem
and proved the existence of solutions
among surfaces of all topological types.

He considered the family of all compact sets $\Sigma \subset \mathbb{R}^n$
with $B \subset \Sigma$
such that $[B] \to 0$ under the induced map

$$H_p(B, \Lambda) \to H_p(\Sigma, \Lambda)$$
on Čech homology with coefficients in $\Lambda$ (say, $\mathbb{Z}$ or $\mathbb{Z}_2$),
Reifenberg 1960

took a radically different approach to the Plateau Problem
and proved the existence of solutions
among surfaces of all topological types.

He considered the family of all compact sets $\Sigma \subset \mathbb{R}^n$
with $B \subset \Sigma$
such that $[B] \to 0$ under the induced map

$$H_p(B, \Lambda) \to H_p(\Sigma, \Lambda)$$

on Čech homology with coefficients in $\Lambda$ (say, $\mathbb{Z}$ or $\mathbb{Z}_2$), and then

he minimized $\mathcal{H}^p(\Sigma)$ in this class.
Reifenberg proved

If $\Sigma$ is one of his solutions to this problem,
Reifenberg proved

If $\Sigma$ is one of his solutions to this problem, there is a (relatively) open dense subset $\Sigma_{\text{reg}} \subset \Sigma$ which is a real analytic submanifold of $\mathbb{R}^n$. 
Reifenberg proved

If $\Sigma$ is one of his solutions to this problem, there is a (relatively) open dense subset $\Sigma_{\text{reg}} \subset \Sigma$ which is a real analytic submanifold of $\mathbb{R}^n$.

Furthermore, if $n = 3$ and $\Lambda = \mathbb{Z}_2$,
Reifenberg proved

If $\Sigma$ is one of his solutions to this problem, there is a (relatively) open dense subset $\Sigma_{\text{reg}} \subset \Sigma$ which is a real analytic submanifold of $\mathbb{R}^n$.

Furthermore, if $n = 3$ and $\Lambda = \mathbb{Z}_2$, all of $\Sigma - B$ is a real analytic submanifold of $\mathbb{R}^n$. 

Reifenberg proved

If $\Sigma$ is one of his solutions to this problem, there is a (relatively) open dense subset $\Sigma_{\text{reg}} \subset \Sigma$ which is a \textbf{real analytic submanifold} of $\mathbb{R}^n$.

Furthermore, if $n = 3$ and $\Lambda = \mathbb{Z}_2$, all of $\Sigma - B$ is a \textbf{real analytic submanifold} of $\mathbb{R}^n$.

A \textbf{complete solution} to the \textbf{unoriented Plateau Problem} in $\mathbb{R}^3$. 


Federer and Fleming – 1960

Federer and Fleming – 1960

wrote an important foundational paper:

Normal and Integral Currents, Ann. of Math.
Federer and Fleming – 1960

wrote an important foundational paper: 

which, among many other things, established the existence of solutions to the Plateau problem in a very general setting.
Federer and Fleming – 1960

wrote an important foundational paper:


which, among many other things, established the existence of solutions to the Plateau problem in a very general setting.

Fundamental to this is the notion of an **Oriented $p$-Rectifiable Set**.

**Federer and Fleming – 1960**

wrote an important foundational paper: *Normal and Integral Currents*, Ann. of Math.
which, among many other things, established the **existence of solutions to the Plateau problem**
in a very general setting.

Fundamental to this is the notion of an **Oriented $p$-Rectifiable Set**.

which leads to the notion of a **Rectifiable $p$-Current**.
Rectifiable Sets.

There are many characterizations.
There are many characterizations.

**Definition.** Let $E$ be an $\mathcal{H}^p$-measurable subset of a riemannian manifold $X$. 
Rectifiable Sets.

There are many characterizations.

**Definition.** Let $E$ be an $\mathcal{H}^p$-measurable subset of a riemannian manifold $X$. Then $E$ is $p$-rectifiable if for every $\epsilon > 0$ there exists a $p$-dimensional embedded $C^1$-submanifold $M \subset X$ with

$$\mathcal{H}^p(E \Delta M) < \epsilon.$$
Rectifiable Sets.

There are many characterizations.

Definition. Let $E$ be an $\mathcal{H}^p$-measurable subset of a riemannian manifold $X$. Then $E$ is $p$-rectifiable if for every $\epsilon > 0$ there exists a $p$-dimensional embedded $C^1$-submanifold $M \subset X$ with

$$\mathcal{H}^p(E \Delta M) < \epsilon.$$ 

Alternative Definition.

$$E \subset \bigcup_{k=1}^{\infty} f_k(\mathbb{R}^p)$$
Rectifiable Sets.

There are many characterizations.

Definition. Let $E$ be an $\mathcal{H}^p$-measurable subset of a riemannian manifold $X$. Then $E$ is $p$-rectifiable if for every $\epsilon > 0$ there exists a $p$-dimensional embedded $C^1$-submanifold $M \subset X$ with

$$\mathcal{H}^p(E \Delta M) < \epsilon.$$ 

Alternative Definition.

$$E \subset \bigcup_{k=1}^{\infty} f_k(\mathbb{R}^p)$$

where each

$$f_k : \mathbb{R}^p \to X$$ is a Lipschitz map.
IMPORTANT FACT.

Rectifiable sets have **tangent planes** a.e.
IMPORTANT FACT.

Rectifiable sets have tangent planes a.e.

Definition.  An orientation of a $p$-rectifiable set $E$ is a measurable choice of orientations of the tangent spaces of $E$. 
**IMPORTANT FACT.**
Rectifiable sets have **tangent planes** a.e.

**Definition.** An **orientation** of a $p$-rectifiable set $E$ is a measurable choice of orientations of the tangent spaces of $E$. i.e.,

an $\mathcal{H}^p$-measurable field of unit simple $p$-vectors $\vec{E}$ with

$$\vec{E}_x \cong T_x E \quad \text{for } \mathcal{H}^p\text{-a.a. } x \in E$$
IMPORTANT CONSEQUENCE.

One can integrate differential forms.
IMPORTANT CONSEQUENCE.

One can integrate differential forms.

Let $E \subset X$ be an oriented $p$-rectifiable set with $\mathcal{H}^p(E) < \infty$. 
IMPORTANT CONSEQUENCE.

One can integrate differential forms.

Let $E \subset X$ be an oriented $p$-rectifiable set with $\mathcal{H}^p(E) < \infty$.

Then for every (smooth) differential $p$-form on $X$

$$\alpha \in \mathcal{E}^p(X)$$
IMPORTANT CONSEQUENCE.

One can integrate differential forms.

Let \( E \subset X \) be an oriented \( p \)-rectifiable set with \( \mathcal{H}^p(E) < \infty \).

Then for every (smooth) differential \( p \)-form on \( X \)

\[
\alpha \in \mathcal{E}^p(X)
\]

The integral

\[
\int_E \alpha \equiv \int_E \alpha \left( \overrightarrow{E_x} \right) \ d\mathcal{H}^p(x)
\]

is well defined.
Rectifiable Sets.

NEXT IMPORTANT CONSEQUENCE.
NEXT IMPORTANT CONSEQUENCE.

We have an embedding.

\[ \{ \text{oriented } p \text{-rectifiable sets} \} \subset \mathcal{E}_p(X) \equiv (\mathcal{E}^p(X))' \]
Next Important Consequence.

We have an embedding.

\{\text{oriented } p\text{-rectifiable sets} \} \subset \mathcal{E}_p(X) \equiv (\mathcal{E}^p(X))'

into the space of \textit{p-dimensional currents} on \(X\).
Rectifiable Sets.

NEXT IMPORTANT CONSEQUENCE.

We have an embedding.

\{\text{oriented } p\text{-rectifiable sets}\} \subset \mathcal{E}_p(X) \equiv (\mathcal{E}^p(X))' \subset \mathcal{E}(X)

into the space of \textit{p-dimensional currents} on \(X\).

HENCE THERE IS A WELL-DEFINED NOTION OF BOUNDARY.
NEXT IMPORTANT CONSEQUENCE.

We have an embedding.

\[ \{ \text{oriented } p\text{-rectifiable sets} \} \subset \mathcal{E}_p(X) \equiv (\mathcal{E}^p(X))' \]

into the space of \textit{p-dimensional currents} on \( X \).

HENCE THERE IS A WELL-DEFINED NOTION OF BOUNDARY.

\[ (\partial[E])(\alpha) \equiv [E](d\alpha) \]
Rectifiable Currents.

Definition. A rectifiable $p$-current is a sum

\[ T = \sum_{j=1}^{\infty} n_j [E_j] \]

where \( \{E_j\}_j \) is a family of disjoint oriented $p$-rectifiable sets,
Rectifiable Currents.

Definition. A rectifiable $p$-current is a sum

$$T = \sum_{j=1}^{\infty} n_j [E_j]$$

where \( \{E_j\}_j \) is a family of disjoint oriented $p$-rectifiable sets, $n_j \in \mathbb{Z}^+$.
Rectifiable Currents.

Definition. A **rectifiable $p$-current** is a sum

\[
T = \sum_{j=1}^{\infty} n_j [E_j]
\]

where \(\{E_j\}_j\) is a family of disjoint oriented $p$-rectifiable sets, \(n_j \in \mathbb{Z}^+\)

\[
\bigcup_{j=1}^{\infty} E_j \subset \subset X
\]
Definition. A **rectifiable $p$-current** is a sum

\[
T = \sum_{j=1}^{\infty} n_j [E_j]
\]

where \(\{E_j\}_j\) is a family of disjoint oriented $p$-rectifiable sets, \(n_j \in \mathbb{Z}^+\)

\[
\bigcup_{j=1}^{\infty} E_j \subseteq X
\]

and the **mass** of $T$

\[
M(T) \equiv \sum_{j=1}^{\infty} n_j \mathcal{H}^p(E_j) < \infty.
\]
Note that we consider any such $T$ to be a current $T \in \mathcal{E}_p(X)$.
Integral Currents.

**Note that** We consider any such $T$ to be a current

$$T \in \mathcal{E}_p(X)$$

and as such it has a boundary

$$\partial T \in \mathcal{E}_{p-1}(X)$$
Integral Currents.

Note that we consider any such $T$ to be a current

$$T \in \mathcal{E}_p(X)$$

and as such it has a boundary

$$\partial T \in \mathcal{E}_{p-1}(X) \quad \text{given by} \quad (\partial T)(\beta) \equiv T(d\beta).$$
Integral Currents.

Note that We consider any such $T$ to be a current

$$T \in \mathcal{E}_p(X)$$

and as such it has a boundary

$$\partial T \in \mathcal{E}_{p-1}(X) \quad \text{given by} \quad (\partial T)(\beta) \equiv T(d\beta).$$

Definition. An integral $p$-current is a current $T \in \mathcal{E}_p(X)$ such that

both $T$ and $\partial T$ are rectifiable.
Integral Currents.

Note that we consider any such $T$ to be a current

$$T \in \mathcal{E}_p(X)$$

and as such it has a boundary

$$\partial T \in \mathcal{E}_{p-1}(X) \quad \text{given by} \quad (\partial T)(\beta) \equiv T(d\beta).$$

Definition. An integral $p$-current is a current $T \in \mathcal{E}_p(X)$ such that

both $T$ and $\partial T$ are rectifiable.

$$\mathcal{I}_p(X) \equiv \{\text{rectifiable } p\text{-currents on } X\}.$$
Integral Currents.

Note that We consider any such $T$ to be a current

$$T \in \mathcal{E}_p(X)$$

and as such it has a boundary

$$\partial T \in \mathcal{E}_{p-1}(X) \quad \text{given by} \quad (\partial T)(\beta) \equiv T(d\beta).$$

**Definition.** An **integral $p$-current** is a current $T \in \mathcal{E}_p(X)$ such that

both $T$ and $\partial T$ are rectifiable.

$$\mathcal{I}_p(X) \equiv \{\text{rectifiable } p\text{-currents on } X\}.$$ 

$$\partial : \mathcal{I}_p(X) \to \mathcal{I}_{p-1}(X) \quad \text{and} \quad \partial^2 = 0.$$
Homology.

For any manifold $X$ we have a chain complex

$$I_n(X) \xrightarrow{\partial} I_{n-1}(X) \xrightarrow{\partial} \cdots \xrightarrow{\partial} I_0(X)$$

Theorem (Federer-Fleming)

There is an equivalence of functors

$$H^* (I^* (X), \partial) \cong H^* (X; \mathbb{Z})$$

This remains true for much more general spaces $X$. 

Homology.

For any manifold $X$ we have a chain complex

$$
\mathcal{I}_n(X) \xrightarrow{\partial} \mathcal{I}_{n-1}(X) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{I}_0(X)
$$

**Theorem (Federer-Fleming)**  
*There is an equivalence of functors*

$$
H_*(\mathcal{I}_*(X), \partial) \cong H_*(X; \mathbb{Z}).
$$
Homology.

For any manifold $X$ we have a chain complex

$$
\mathcal{I}_n(X) \xrightarrow{\partial} \mathcal{I}_{n-1}(X) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{I}_0(X)
$$

**Theorem (Federer-Fleming)** *There is an equivalence of functors*

$$
H_\ast(\mathcal{I}_\ast(X), \partial) \cong H_\ast(X; \mathbb{Z}).
$$

This remains true for much more general spaces $X$. 
Compactness.

Given a compact set $K \subset X$ and $c > 0$, let

$$I_p(X)_{K,c} \subset I_p(X)$$
Compactness.

Given a compact set $K \subset X$ and $c > 0$, let

$$\mathcal{I}_p(X)_{K,c} \subset \mathcal{I}_p(X)$$

be the set of $T$ with

$$M(T) \leq c, \quad M(\partial T) \leq c \quad \text{and} \quad \text{supp}(T) \subset K.$$
Compactness.

Given a compact set $K \subset X$ and $c > 0$, let

$$\mathcal{I}_p(X)_{K,c} \subset \mathcal{I}_p(X)$$

be the set of $T$ with

$$M(T) \leq c, \quad M(\partial T) \leq c \quad \text{and} \quad \text{supp}(T) \subset K.$$

**Theorem (Federer-Fleming)**

*The set $\mathcal{I}_p(X)_{K,c}$ is compact in the weak topology.*
The Plateau Problem in $X$.

Since mass is lower semi-continuous in the weak topology, the compactness result solves a very general form of the Plateau Problem.
The Plateau Problem in $X$.

Since mass is lower semi-continuous in the weak topology, the compactness result solves a very general form of the Plateau Problem. Let $X$ be compact riemannian or, say, $\mathbb{R}^n$. 
The Plateau Problem in $X$.

Since mass is lower semi-continuous in the weak topology, the compactness result solves a very general form of the Plateau Problem. Let $X$ be compact riemannian or, say, $\mathbb{R}^n$.

**Theorem (Federer-Fleming)**

Let $B \in \mathcal{I}_{p-1}(X)$ be a cycle ($\partial B = 0$)
The Plateau Problem in $X$.

Since mass is lower semi-continuous in the weak topology, the compactness result solves a very general form of the Plateau Problem. Let $X$ be compact riemannian or, say, $\mathbb{R}^n$.

**Theorem (Federer-Fleming)**

Let $B \in \mathcal{I}_{p-1}(X)$ be a cycle ($\partial B = 0$) such that $B = \partial T_0$ for some $T_0 \in \mathcal{I}_p(X)$.
The Plateau Problem in $X$.

Since mass is lower semi-continuous in the weak topology, the compactness result solves a very general form of the Plateau Problem. Let $X$ be compact riemannian or, say, $\mathbb{R}^n$.

**Theorem (Federer-Fleming)**

Let $B \in \mathcal{I}_{p-1}(X)$ be a cycle ($\partial B = 0$) such that $B = \partial T_0$ for some $T_0 \in \mathcal{I}_p(X)$.

Then there exists $T \in \mathcal{I}_p(X)$ with $T - T_0 = dR_0$ some $R_0 \in \mathcal{I}_{p+1}(X)$. 

Blaine Lawson
Jim Simons' Work on Minimal Varieties
May 24, 2013
The Plateau Problem in $X$.

Since mass is lower semi-continuous in the weak topology, the compactness result solves a very general form of the Plateau Problem. Let $X$ be compact riemannian or, say, $\mathbb{R}^n$.

**Theorem (Federer-Fleming)**

Let $B \in \mathcal{I}_{p-1}(X)$ be a cycle ($\partial B = 0$) such that $B = \partial T_0$ for some $T_0 \in \mathcal{I}_p(X)$.

Then there exists $T \in \mathcal{I}_p(X)$ with $T - T_0 = dR_0$ some $R_0 \in \mathcal{I}_{p+1}(X)$ such that

$$M(T) = \inf_{R \in \mathcal{I}_{p+1}(X)} M(T + \partial R)$$
Picture.
Consequence.

Let \( X \) be a compact riemannian manifold.

**Corollary (Federer-Fleming)**

\[ \text{Every homology class } u \in H_p(X; \mathbb{Z}) \text{ contains an integral current of least mass.} \]
Regularity Theory.

Fleming 1962

Theorem. (Fleming).

Let $T \in I_2(R^3)$ be a current of least mass (among all integral currents with the same boundary).

Then $\text{supp}(T) - \text{supp} \partial T$ is a regular minimal surface in $R^3 - \text{supp} \partial T$.

This solves the oriented Plateau Problem in $R^3$ among surfaces of all topological types.

The result holds in general Riemannian 3-manifolds.
Fleming 1962

Combined his work with Federer, techniques of Reifenberg, and clever arguments to prove

Theorem. (Fleming).

Let $T \in I_2(\mathbb{R}^3)$ be a current of least mass (among all integral currents with the same boundary).

Then $\text{supp}(T) - \text{supp} \partial T$ is a regular minimal surface in $\mathbb{R}^3 - \text{supp} \partial T$.

This solves the oriented Plateau Problem in $\mathbb{R}^3$ among surfaces of all topological types.

The result holds in general riemannian 3-manifolds.

Regularity Theory.
Regularity Theory.

**Fleming 1962**

Combined his work with Federer, techniques of Reifenberg, and clever arguments to prove

the complete interior regularity of mass-minimizing integral currents in $\mathbb{R}^3$. 

**Theorem.** (Fleming).

Let $T \in I_2(\mathbb{R}^3)$ be a current of least mass (among all integral currents with the same boundary).

Then $\text{supp}(T) - \text{supp}\partial T$ is a regular minimal surface in $\mathbb{R}^3 - \text{supp}\partial T$. This solves the oriented Plateau Problem in $\mathbb{R}^3$ among surfaces of all topological types.

The result holds in general riemannian 3-manifolds.
Regularity Theory.

**Fleming  1962**

Combined his work with Federer, techniques of Reifenberg, and clever arguments to prove

the complete interior regularity of mass-minimizing integral currents in $\mathbb{R}^3$.

**Theorem. (Fleming).** Let $T \in \mathcal{I}_2(\mathbb{R}^3)$ be a current of least mass (among all integral currents with the same boundary).
Regularity Theory.

Fleming 1962

Combined his work with Federer, techniques of Reifenberg, and clever arguments to prove

the complete interior regularity of mass-minimizing integral currents in $\mathbb{R}^3$.

**Theorem. (Fleming).** Let $T \in I_2(\mathbb{R}^3)$ be a current of least mass (among all integral currents with the same boundary). Then

$$\text{supp}(T) - \text{supp}\partial T$$

**is a regular minimal surface in** $\mathbb{R}^3 - \text{supp}\partial T$. 

This solves the oriented Plateau Problem in $\mathbb{R}^3$ among surfaces of all topological types. The result holds in general Riemannian 3-manifolds.
Regularity Theory.

**Fleming 1962**

Combined his work with Federer, techniques of Reifenberg, and clever arguments to prove

the complete interior regularity of mass-minimizing integral currents in \( \mathbb{R}^3 \).

**Theorem. (Fleming).** Let \( T \in \mathcal{I}_2(\mathbb{R}^3) \) be a current of least mass (among all integral currents with the same boundary). Then

\[
\text{supp}(T) - \text{supp}\partial T \text{ is a regular minimal surface in } \mathbb{R}^3 - \text{supp}\partial T.
\]

This solves the **oriented** Plateau Problem in \( \mathbb{R}^3 \) among surfaces of **all topological types**.
Regularity Theory.

**Fleming 1962**

Combined his work with Federer, techniques of Reifenberg, and clever arguments to prove

the complete interior regularity of mass-minimizing integral currents in $\mathbb{R}^3$.

**Theorem. (Fleming).** Let $T \in \mathcal{I}_2(\mathbb{R}^3)$ be a current of least mass (among all integral currents with the same boundary). Then  

$$\text{supp}(T) - \text{supp}\partial T$$  

is a regular minimal surface in $\mathbb{R}^3 - \text{supp}\partial T$.

This solves the **oriented** Plateau Problem in $\mathbb{R}^3$ among surfaces of all topological types.

The result holds in general riemannian 3-manifolds.
Cautionary Note: Multiplicities.

A mass-minimizing integral current $T$ can have integer multiplicities
Cautionary Note: Multiplicities.

A mass-minimizing integral current $T$ can have integer multiplicities on each connected component of its support.
Cautionary Note: Multiplicities.

A mass-minimizing integral current $T$ can have integer multiplicities on each connected component of its support.

So $T$ has the form of a locally finite sum

$$T = \sum_\limits{k} n_k [M_k].$$
Higher Dimensions and Codimensions??.

Complete regularity fails for codimension $\geq 2$. 

Theorem. (Federer). Let $V$ be a complex analytic subvariety, $\dim C(V) = p$ in a Kähler manifold $X$, (for example a domain in $C^n$). Then $V$ defines an integral current $[V] \in I_{loc}^{2p}(X)$ with $\partial [V] = 0$ which is homologically mass-minimizing in $X$. That is, for every open set $\Omega \subset \subset X$, $M([V]) \leq M([V] + \partial S)$ $\forall S \in I_{2p+1}(\Omega)$.
Higher Dimensions and Codimensions??.

Complete regularity fails for codimension $\geq 2$.

**Theorem. (Federer).** Let $V$ be a complex analytic subvariety, $\dim_C(V) = p$ in a Kähler manifold $X$,
Complete regularity fails for codimension $\geq 2$.

Theorem. (Federer). Let $V$ be a complex analytic subvariety, $\dim_C(V) = p$ in a Kähler manifold $X$, (for example a domain in $\mathbb{C}^n$).
Complete regularity fails for codimension $\geq 2$.

**Theorem. (Federer).** Let $V$ be a complex analytic subvariety, $\dim_{\mathbb{C}}(V) = p$ in a Kähler manifold $X$, (for example a domain in $\mathbb{C}^n$). Then $V$ defines an integral current $[V] \in \mathcal{I}_{2p}(X)$ with $\partial [V] = 0$

which is **homologically mass-minimizing** in $X$. 


Complete regularity fails for codimension \( \geq 2 \).

**Theorem. (Federer).** Let \( V \) be a complex analytic subvariety, \( \dim_{\mathbb{C}}(V) = p \) in a Kähler manifold \( X \), (for example a domain in \( \mathbb{C}^n \)). Then \( V \) defines an integral current

\[
[V] \in \mathcal{I}_{2p}^{\text{loc}}(X) \quad \text{with} \quad \partial[V] = 0
\]

which is **homologically mass-minimizing** in \( X \).

That is, for every open set \( \Omega \subset X \)

\[
M([V]) \leq M([V] + \partial S) \quad \forall \ S \in \mathcal{I}_{2p+1}(\Omega)
\]
Regularity in Codimension-One.

Work of Almgren and De Giorgi.
Regularity in Codimension-One.

Work of Almgren and De Giorgi.

Theorem. (Almgren).

*Complete interior regularity holds for mass-minimizing integral currents of dimension 3 in 4-manifolds.*
Regularity in Codimension-One.

Work of Almgren and De Giorgi.

Theorem. (Almgren).

*Complete interior regularity holds for mass-minimizing integral currents of dimension 3 in 4-manifolds.*

Theorem. (Almgren-De Giorgi).

*The Bernstein conjecture holds for minimal graphs* $\Gamma = \{ (x, f(x)) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n \}$ *for* $n \leq 4$. 
The Proof – Revolves Around Cones.

\[ S_n \equiv \{ x \in \mathbb{R}^n_{+1} : \|x\| = 1 \} \]

and consider a compact submanifold \( M_p \subset S_n \).

The cone on \( M_p \) is the set \( C(M_p) = \{ tx \in \mathbb{R}^n_{+1} : x \in M_p \text{ and } t \geq 0 \} \).

This concept extends naturally to currents.
The Proof – Revolves Around Cones.

Set

\[ S^n \equiv \{ x \in \mathbb{R}^{n+1} : \| x \| = 1 \} \]

and consider a compact submanifold

\[ M^p \subset S^n. \]
The Proof – Revolves Around Cones.

Set

$$S^n \equiv \{ x \in \mathbb{R}^{n+1} : \|x\| = 1 \}.$$ 

and consider a compact submanifold

$$M^p \subset S^n.$$ 

The cone on $M^p$ is the set

$$C(M^p) = \{ tx \in \mathbb{R}^{n+1} : x \in M^p \text{ and } t \geq 0 \}.$$
The Proof – Revolves Around Cones.

Set
\[ S^n \equiv \{ x \in \mathbb{R}^{n+1} : \|x\| = 1 \}. \]

and consider a compact submanifold
\[ M^p \subset S^n. \]

The cone on \( M^p \) is the set
\[ C(M^p) = \{ tx \in \mathbb{R}^{n+1} : x \in M^p \text{ and } t \geq 0 \} \]
The Proof – Revolves Around Cones.

Set

\[ S^n \equiv \{ x \in \mathbb{R}^{n+1} : \| x \| = 1 \} . \]

and consider a compact submanifold

\[ M^p \subset S^n . \]

The cone on \( M^p \) is the set

\[ C(M^p) = \{ tx \in \mathbb{R}^{n+1} : x \in M^p \text{ and } t \geq 0 \} . \]

This concept extends naturally to currents.
Proposition.

Suppose that $C = C(M) \in I_{\text{loc}}^{+1}(R^n_{+1})$ is the cone on a current $M \in I_p(S_n)$. Then $C(M)$ is minimal in $R^n_{+1} \iff M$ is minimal in $S_n$. 

QUESTION: $C(M)$ is minimizing stable in $R^n_{+1} \iff M$ is ??? in $S_n$. 

Blaine Lawson

Jim Simons' Work on Minimal Varieties

May 24, 2013
Proposition.

Suppose that

\[ C = C(M) \in \mathcal{I}_{p+1}^{\text{loc}}(\mathbb{R}^{n+1}) \]

is the **cone on a current**

\[ M \in \mathcal{I}_p(S^n). \]
Proposition.

Suppose that

\[ C = C(M) \in I_{p+1}^{loc}(\mathbb{R}^{n+1}) \]

is the cone on a current

\[ M \in I_p(S^n). \]

Then

\[ C(M) \text{ is minimal in } \mathbb{R}^{n+1} \iff M \text{ is minimal in } S^n \]
Proposition.

Suppose that

\[ C = C(M) \in I_{p+1}^{\text{loc}}(\mathbb{R}^{n+1}) \]

is the cone on a current

\[ M \in I_p(S^n). \]

Then

\[ C(M) \text{ is minimal in } \mathbb{R}^{n+1} \iff M \text{ is minimal in } S^n. \]

QUESTION:

\[ C(M) \text{ is } \begin{cases} \text{minimizing} \\ \text{stable} \end{cases} \text{ in } \mathbb{R}^{n+1} \iff M \text{ is } ??? \text{ in } S^n. \]
How Do Cones Enter?

Suppose $T$ is a mass-minimizing integral current. Then at each $x \in \text{supp}(T)$ the current $T$ has tangent cones which are mass-minimizing.

IDEA: Consider sequences of dilations.
How Do Cones Enter?

Suppose $T$ is a mass-minimizing integral current
How Do Cones Enter?

Suppose $T$ is a mass-minimizing integral current.

Then at each $x \in \text{supp}(T)$ the current $T$ has tangent cones which are mass-minimizing.
How Do Cones Enter?

Suppose $T$ is a mass-minimizing integral current.

Then at each $x \in \text{supp}(T)$ the current $T$ has tangent cones which are mass-minimizing.

IDEA: Consider sequences of dilations.
How Do Cones Enter?

Suppose $\Gamma$ is a minimal graph of codimension-1 in $\mathbb{R}^n$. Then $\Gamma$ is a mass-minimizing integral current. If the graphing function is defined over all of $\mathbb{R}^n$, then we can produce cones which are mass-minimizing in $\mathbb{R}^n$. They are of the form $C^0 \times \mathbb{R}$. 

IDEA: Consider sequences of contractions.
How Do Cones Enter?

Suppose $\Gamma$ is a minimal graph of codimension-1 in $\mathbb{R}^n$. Then $\Gamma$ is a mass-minimizing integral current.
How Do Cones Enter?

Suppose $\Gamma$ is a minimal graph of codimension-1 in $\mathbb{R}^n$.
Then $\Gamma$ is a mass-minimizing integral current.
If the graphing function is defined over all of $\mathbb{R}^n$, 

$C_0 \times \mathbb{R}$. 

**IDEA:** Consider sequences of contractions.
How Do Cones Enter?

Suppose $\Gamma$ is a minimal graph of codimension-1 in $\mathbb{R}^n$. Then $\Gamma$ is a mass-minimizing integral current. If the graphing function is defined over all of $\mathbb{R}^n$, then we can produce cones which are mass-minimizing in $\mathbb{R}^n$. 
How Do Cones Enter?

Suppose $\Gamma$ is a minimal graph of codimension-1 in $\mathbb{R}^n$. Then $\Gamma$ is a mass-minimizing integral current. If the graphing function is defined over all of $\mathbb{R}^n$, then we can produce cones which are mass-minimizing in $\mathbb{R}^n$. They are of the form $C_0 \times \mathbb{R}$.

IDEA: Consider sequences of contractions.
How Do Cones Enter?

Suppose $\Gamma$ is a minimal graph of codimension-1 in $\mathbb{R}^n$. Then $\Gamma$ is a mass-minimizing integral current. If the graphing function is defined over all of $\mathbb{R}^n$, then we can produce cones which are mass-minimizing in $\mathbb{R}^n$. They are of the form $C_0 \times \mathbb{R}$.

IDEA: Consider sequences of contractions.
The Key Idea:

A tangent cone to a cone $C$ (at a point $x$ away from the vertex) splits as a product $T_x C = \mathbb{R} \times C_0$ where $\dim(C_0) = \dim(C) - 1$. 

One can now apply induction on dimension.
The Key Idea:

A tangent cone to a cone $C$
The Key Idea:

A tangent cone to a cone $C$

(at a point $x$ away from the vertex)
The Key Idea:

A tangent cone to a cone $C$
(at a point $x$ away from the vertex)
splits as a product

$$T_x C = \mathbb{R} \times C_0$$
The Key Idea:

A tangent cone to a cone $C$
(at a point $x$ away from the vertex)
splits as a product

\[ T_x C = \mathbb{R} \times C_0 \]

where

\[ \dim(C_0) = \dim(C) - 1. \]
The Key Idea:

A tangent cone to a cone $C$
(at a point $x$ away from the vertex)
splits as a product

$$T_x C = \mathbb{R} \times C_0$$

where

$$\dim(C_0) = \dim(C) - 1.$$ 

One can now apply induction on dimension
The Key Induction Step:

Work of the previous people shows:

If interior regularity holds for minimizing hypersurfaces in dimension $n$, then every minimizing cone in $\mathbb{R}^{n+1}$ is the cone on a regular minimal submanifold $M \subset S^n$. 

Blaine Lawson
Jim Simons’ Work on Minimal Varieties
May 24, 2013 53 / 2
The Key Induction Step:

Work of the previous people shows:

If interior regularity holds for minimizing hypersurfaces in dimension $n$, then every minimizing cone in $\mathbb{R}^{n+1}$ is the cone on a regular minimal submanifold $M \subset S^n$. 
The Key Induction Step:

Work of the previous people shows:

If interior regularity holds for minimizing hypersurfaces in dimension $n$, then every minimizing cone in $\mathbb{R}^{n+1}$ is the cone on a regular minimal submanifold $M \subset S^n$. 
The Key Induction Step:

Work of the previous people shows:

If interior regularity holds for minimizing hypersurfaces in dimension $n$, then every minimizing cone in $\mathbb{R}^{n+1}$ is
The Key Induction Step:

Work of the previous people shows:

If interior regularity holds for minimizing hypersurfaces in dimension $n$, then every minimizing cone in $\mathbb{R}^{n+1}$ is the cone on an regular minimal submanifold $M \subset S^n$. 
Simons’ Theorem:

\[ \text{THEOREM. (J. Simons 1968)} \]

Suppose \( M^{n-1} \subset S^n \) is a compact minimal hypersurface such that the cone \( C(M) \subset \mathbb{R}^{n+1} \) is stable (for example, mass-minimizing). Then \( M^{n-1} \) is a totally geodesic hypersphere \( S^{n-1} \subset S^n \), that is, \( C(M) = \mathbb{R}^n \subset \mathbb{R}^{n+1} \) is a linear subspace provided \( n+1 \leq 7 \).

FURTHERMORE, this assertion is false if \( n+1 \geq 8 \).
Simons’ Theorem:

**THEOREM. (J. Simons 1968)** Suppose $M^{n-1} \subset S^n$ is a compact minimal hypersurface such that the cone

$$C(M) \subset \mathbb{R}^{n+1}$$

is stable (for example, mass-minimizing).
**Simons’ Theorem:**

**THEOREM. (J. Simons 1968)** Suppose $M^{n-1} \subset S^n$ is a compact minimal hypersurface such that the cone

$$C(M) \subset \mathbb{R}^{n+1}$$

is stable (for example, mass-minimizing). Then $M^{n-1}$ is a totally geodesic hypersphere $S^{n-1} \subset S^n$, provided $n+1 \leq 7$. **Furthermore,** this assertion is false if $n+1 \geq 8$. 
Simons’ Theorem:

**THEOREM. (J. Simons 1968)** Suppose $M^{n-1} \subset S^n$ is a compact minimal hypersurface such that the cone

\[ C(M) \subset \mathbb{R}^{n+1} \]

is **stable** (for example, mass-minimizing). Then $M^{n-1}$ is a totally geodesic hypersphere $S^{n-1} \subset S^n$, that is,

\[ C(M) = \mathbb{R}^n \subset \mathbb{R}^{n+1} \]

is a linear subspace
Simons’ Theorem:

**THEOREM. (J. Simons 1968)** Suppose $M^{n-1} \subset S^n$ is a compact minimal hypersurface such that the cone

$$C(M) \subset \mathbb{R}^{n+1}$$

is **stable** (for example, mass-minimizing). Then $M^{n-1}$ is a totally geodesic hypersphere $S^{n-1} \subset S^n$, that is,

$$C(M) = \mathbb{R}^n \subset \mathbb{R}^{n+1}$$

is a linear subspace

provided $n + 1 \leq 7$. 

Simons’ Theorem:

THEOREM. (J. Simons 1968) Suppose $M^{n-1} \subset S^n$ is a compact minimal hypersurface such that the cone

$$C(M) \subset \mathbb{R}^{n+1}$$

is stable (for example, mass-minimizing). Then $M^{n-1}$ is a totally geodesic hypersphere $S^{n-1} \subset S^n$, that is,

$$C(M) = \mathbb{R}^n \subset \mathbb{R}^{n+1}$$

is a linear subspace

provided $n + 1 \leq 7$.

FURTHERMORE,

This assertion is false if $n + 1 \geq 8$.
Corollaries:

*Complete interior regularity holds for mass-minimizing integral currents of codimension-one in Riemannian manifolds of dimension \( \leq 7 \).*
Corollaries:

Complete interior regularity holds for mass-minimizing integral currents of codimension-one in Riemannian manifolds of dimension \( \leq 7 \).

The Bernstein Conjecture holds for minimal graphs \( \{ x_{n+1} = f(x_1, \ldots, x_n) \} \) when \( n \leq 7 \).
Simons’ Example:

\[ C(S^3 \times S^3) \equiv \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x| = |y| \} \subset \mathbb{R}^8 \]
Simons’ Example:

\[ C(S^3 \times S^3) \equiv \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x| = |y|\} \subset \mathbb{R}^8 \]

**Theorem.** (Bombieri, De Giorgi, Giusti (1969))

*Simons’ cone is mass-minimizing in \( \mathbb{R}^8 \).*
Simons’ Example:

\[ C(S^3 \times S^3) \equiv \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x| = |y|\} \subset \mathbb{R}^8 \]

**Theorem.** (Bombieri, De Giorgi, Giusti (1969))

Simons’ cone is mass-minimizing in \( \mathbb{R}^8 \).

Hence, interior regularity fails in all dimensions \( \geq 8 \).
Simons’ Example:

\[ C(S^3 \times S^3) \equiv \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x| = |y|\} \subset \mathbb{R}^8 \]

Theorem. (Bombieri, De Giorgi, Giusti (1969))

Simons’ cone is mass-minimizing in \( \mathbb{R}^8 \).

Hence, interior regularity fails in all dimensions \( \geq 8 \).

The Bernstein Conjecture is false for all \( n \geq 8 \).
Further Results of Jim Simons.

Some Differential Geometry

Let $M \subset X$ be a submanifold of a riemannian manifold $X$ with Levi-Civita connection $\nabla$. For vector fields $V, W$ on $M$

$$\nabla V W = (\nabla V W)^T + (\nabla V W)^N$$

Then $\nabla V W \equiv (\nabla V W)^T$ is the Levi-Civita connection of the induced riemannian metric on $M$ and $B_{V, W} \equiv (\nabla V W)^N$ is the Second Fundamental Form of $M$ in $X$. 
Some Differential Geometry

Let $M \subset X$ be a submanifold of a riemannian manifold $X$ with Levi-Civita connection $\nabla$. Then $\nabla V W \equiv (\nabla V W)_T$ is the Levi-Civita connection of the induced riemannian metric on $M$ and $B V W \equiv (\nabla V W)_N$ is the Second Fundamental Form of $M$ in $X$. 
Further Results of Jim Simons.

Some Differential Geometry

Let $M \subset X$ be a submanifold of a Riemannian manifold $X$ with Levi-Civita connection $\nabla$.

For vector fields $V, W$ on $M$

$$\nabla_V W = (\nabla_V W)^T + (\nabla_V W)^N$$

Then $\nabla_V W$ is the Levi-Civita connection of the induced Riemannian metric on $M$ and $B_{V, W} = (\nabla_V W)^N$ is the Second Fundamental Form of $M$ in $X$. 

Blaine Lawson
Jim Simons' Work on Minimal Varieties
May 24, 2013 57 / 2
Some Differential Geometry

Let $M \subset X$ be a submanifold of a riemannian manifold $X$ with Levi-Civita connection $\nabla$.

For vector fields $V, W$ on $M$

\[ \nabla_V W = (\nabla_V W)^T + (\nabla_V W)^N \]

Then $\nabla_V W \equiv (\nabla_V W)^T$ is the Levi-Civita connection of the induced riemannian metric on $M$. 

Some Differential Geometry

Let $M \subset X$ be a submanifold of a riemannian manifold $X$ with Levi-Civita connection $\nabla$.

For vector fields $V, W$ on $M$

$$\nabla_V W = (\nabla_V W)^T + (\nabla_V W)^N$$

Then $\nabla_V W \equiv (\nabla_V W)^T$ is the Levi-Civita connection of the induced riemannian metric on $M$ and

$$B_{V, W} \equiv (\nabla_V W)^N$$

is the Second Fundamental Form of $M$ in $X$. 
The Second Fundamental Form.

\[ B_{V,W} \equiv (\nabla_V W)^N \]

is a field of \textbf{symmetric} 2-forms on \( T(M) \)
with values in the normal bundle \( N(M) \)
The Second Fundamental Form.

\[ B_{V \cdot W} \equiv (\nabla_V W)^N \]

is a field of \textbf{symmetric} 2-forms on \( T(M) \) with values in the normal bundle \( N(M) \).

The \textbf{mean curvature vector field} is the normal vector field along \( M \) given by

\[ H \equiv \text{trace}B. \]

**The First Variational Formula** Let \( \varphi_t : M \to X \) be a normal deformation of \( M \) with derivative \( V \) at \( t = 0 \). Then

\[ \left. \frac{\partial}{\partial t} \text{vol} \{ \varphi_t(M) \} \right|_{t=0} = - \int_M \langle H, V \rangle. \]
The Second Fundamental Form.

**The First Variational Formula** Let $\varphi_t : M \to X$ be a normal deformation of $M$ with derivative $V$ at $t = 0$. Then

$$\frac{\partial}{\partial t} \text{vol} \{ \varphi_t(M) \} \bigg|_{t=0} = - \int_M \langle H, V \rangle.$$
The Second Fundamental Form.

**The First Variational Formula** Let \( \varphi_t : M \to X \) be a normal deformation of \( M \) with derivative \( V \) at \( t = 0 \). Then

\[
\frac{\partial}{\partial t} \text{vol} \{ \varphi_t(M) \} \bigg|_{t=0} = - \int_M \langle H, V \rangle.
\]

**The Second Variational Formula** Suppose \( H \equiv 0 \) on \( M \). Then

\[
\frac{\partial^2}{\partial t^2} \text{vol} \{ \varphi_t(M) \} \bigg|_{t=0} = \int_M \langle \nabla^* \nabla V - B(V) + \overline{R}(V), V \rangle
\]
The Second Fundamental Form.

The First Variational Formula Let \( \varphi_t : M \to X \) be a normal deformation of \( M \) with derivative \( V \) at \( t = 0 \). Then

\[
\frac{\partial}{\partial t} \text{vol} \{ \varphi_t(M) \} \bigg|_{t=0} = - \int_M \langle H, V \rangle.
\]

The Second Variational Formula Suppose \( H \equiv 0 \) on \( M \). Then

\[
\frac{\partial^2}{\partial t^2} \text{vol} \{ \varphi_t(M) \} \bigg|_{t=0} = \int_M \langle \nabla^* \nabla V - B(V) + \overline{R}(V), V \rangle
\]

where

\[
B \equiv B \circ B^t : N \to N
\]

(recall \( B : T \otimes T \to N \)).
The Second Fundamental Form.

**The First Variational Formula** Let $\varphi_t : M \to X$ be a normal deformation of $M$ with derivative $V$ at $t = 0$. Then

$$\frac{\partial}{\partial t} \text{vol} \{ \varphi_t(M) \} \bigg|_{t=0} = - \int_M \langle H, V \rangle.$$

**The Second Variational Formula** Suppose $H \equiv 0$ on $M$. Then

$$\frac{\partial^2}{\partial t^2} \text{vol} \{ \varphi_t(M) \} \bigg|_{t=0} = \int_M \langle \nabla^* \nabla V - \mathcal{B}(V) + \overline{R}(V) , V \rangle$$

where

$$\mathcal{B} \equiv B \circ B^t : N \to N$$

(recall $B : T \otimes T \to N$), and

$$\overline{R}(V) = \sum_{j=1}^p \overline{R}_{e_j} V(e_j)$$
Simons Fundamental Equation.

**THEOREM. (First-order system)**

Let $M \subset X$ be a minimal submanifold with second fundamental form $B$. 

\[ \nabla V(B)(W, U) - \nabla W(B)(V, U) = (R V, W) N \]

**Codazzi Equations**

\[ p \sum_{j=1} \nabla e_j(B)(e_j, V) = p \sum_{j=1} (R e_j, V e_j) N \]

**THEOREM. (Second-order equation)**

Let $M \subset X$ be a minimal submanifold with second fundamental form $B$. 

\[ \nabla^\ast \nabla B = F(B, R, \nabla R) \]
Simons Fundamental Equation.

THEOREM. (First-order system)

Let \( M \subset X \) be a minimal submanifold with second fundamental form \( B \).

\[
\nabla_V(B)(W, U) - \nabla_W(B)(V, U) = \left( \overline{R}_V, W U \right)^N
\]

Codazzi Equations

\[
\sum_{j=1}^p \nabla_{e_j}(B)(e_j, V) = \sum_{j=1}^p \left( \overline{R}_{e_j}, V e_j \right)^N
\]
Simons Fundamental Equation.

**THEOREM. (First-order system)**

Let $M \subset X$ be a minimal submanifold with second fundamental form $B$.

$$\nabla_V (B)(W, U) - \nabla_W (B)(V, U) = \left( \overline{R}_{V,W} U \right)^N$$

**Codazzi Equations**

$$\sum_{j=1}^{p} \nabla_{e_j} (B)(e_j, V) = \sum_{j=1}^{p} \left( \overline{R}_{e_j, V} e_j \right)^N$$

**THEOREM. (Second-order equation)**

Let $M \subset X$ be a minimal submanifold with second fundamental form $B$.

$$\nabla^{*} \nabla B = \mathcal{F}(B, \overline{R}, \nabla \overline{R})$$
Simons Fundamental Equation.

$$\nabla^* \nabla B = \mathcal{F}(B, \bar{R}, \nabla \bar{R})$$

had many applications:

- The important stability result above.
- Isolation results: e.g. Suppose $M_n \subset S^{n+1}$ is a minimal submanifold with $\|B\| < n$ pointwise on $M$. Then $M_n - 1 = S_n - 1$ is a totally geodesic "equator.
- Engendered decades of papers on the subject.
Simons Fundamental Equation.

\[\nabla^* \nabla B = \mathcal{F}(B, \bar{R}, \nabla \bar{R})\]

had many applications:

- The important stability result above.
- Isolation results: e.g. Suppose \( M_n \subset S^{n+1} \) is a minimal submanifold with \( \|B\| < n \) pointwise on \( M \).
  Then \( M_\cap S^{n-1} = \) a totally geodesic "equator".
- Engendered decades of papers on the subject.
Simons Fundamental Equation.

\[ \nabla^* \nabla B = \mathcal{F}(B, \bar{R}, \nabla \bar{R}) \]

had many applications:

- The important stability result above.
- Isolation results:

Blaine Lawson
Jim Simons' Work on Minimal Varieties
May 24, 2013 61 / 2
Simons Fundamental Equation.

\[ \nabla^* \nabla B = \mathcal{F}(B, \overline{R}, \nabla \overline{R}) \]

had many applications:

- The important stability result above.
- Isolation results: e.g.

\[
\text{Suppose } M^n \subset S^{n+1} \text{ is a minimal submanifold with } \|B\| < n \text{ pointwise on } M.
\]
Simons Fundamental Equation.

\[ \nabla^* \nabla B = \mathcal{F}(B, \bar{R}, \nabla \bar{R}) \]

had many applications:

- The important stability result above.
- Isolation results: e.g.

Suppose \( M^n \subset S^{n+1} \) is a minimal submanifold with \( \|B\| < n \) pointwise on \( M \).

Then \( M^{n-1} = S^{n-1} \) is a totally geodesic “equator”.

Engendered decades of papers on the subject.
Simons Fundamental Equation.

\[ \nabla^* \nabla B = \mathcal{F}(B, \overline{R}, \nabla \overline{R}) \]

had many applications:

- The important stability result above.
- Isolation results: e.g.

  Suppose \( M^n \subset S^{n+1} \) is a minimal submanifold with
  \[ \|B\| < n \]
  pointwise on \( M \).
  Then \( M^{n-1} = S^{n-1} \) is a totally geodesic “equator”.

- Engendered decades of papers on the subject.
Stable Currents in Projective Space.

Let $\mathcal{H}$ denote the space of holomorphic vector fields on $\mathbb{P}^n(\mathbb{C})$.

Let $S$ be an integral current on $\mathbb{P}^n(\mathbb{C})$.

Define a quadratic form $Q_S$ on $\mathcal{H}$ by

$$Q_S(V) = \lim_{t \to 0} \frac{d^2}{dt^2} M \{ (\phi_t)^* S \}$$

Theorem. (Lawson-Simons)

$$\text{trace}(Q_S) = -\int \langle J(\rightarrow S_x) \rangle \|S\|^2 (x)$$

Corollary (Using Harvey-Shiffman).

Every stable integral current in $\mathbb{P}^n(\mathbb{C})$ is an algebraic cycle.
Stable Currents in Projective Space.

Let $\mathcal{H}$ denote the space of holomorphic vector fields on $\mathbb{P}^n(\mathbb{C})$. Let $S$ be an integral current on $\mathbb{P}^n(\mathbb{C})$. Define a quadratic form $Q_S$ on $\mathcal{H}$ by

$$Q_S(V) = \left. \frac{d^2}{dt^2} \langle \phi_t^* S \rangle \right|_{t=0}.$$ 

Theorem. (Lawson-Simons) \[ \text{trace} \left( Q_S \right) = -\int J(\rightarrow S_x) \parallel S \parallel (x). \]

Corollary (Using Harvey-Shiffman). Every stable integral current in $\mathbb{P}^n(\mathbb{C})$ is an algebraic cycle.
Stable Currents in Projective Space.

Let $\mathcal{H}$ denote the space of holomorphic vector fields on $\mathbb{P}^n(\mathbb{C})$.

Let $S$ be an integral current on $\mathbb{P}^n(\mathbb{C})$,

Define a quadratic form $Q_S$ on $\mathcal{H}$ by

\[
Q_S(V) = \left. \frac{d^2}{dt^2} M(\varphi_t^* S) \right|_{t=0}.
\]
Stable Currents in Projective Space.

Let $\mathcal{H}$ denote the space of holomorphic vector fields on $\mathbb{P}^n(\mathbb{C})$. Let $S$ be an integral current on $\mathbb{P}^n(\mathbb{C})$, Define a quadratic form $Q_S$ on $\mathcal{H}$ by

$$Q_S(V) = \left. \frac{d^2}{dt^2} M \{ (\varphi_t)_* S \} \right|_{t=0}.$$
Stable Currents in Projective Space.

Let $\mathcal{H}$ denote the space of holomorphic vector fields on $\mathbb{P}^n(\mathbb{C})$. Let $S$ be an integral current on $\mathbb{P}^n(\mathbb{C})$.

Define a quadratic form $Q_S$ on $\mathcal{H}$ by

$$Q_S(V) = \left. \frac{d^2}{dt^2} M \{ (\varphi_t)_* S \} \right|_{t=0}.$$ 

**Theorem. (Lawson-Simons)**

$$\text{trace} (Q_S) = - \int \| J(\vec{S}_x) \|^2 \| S \| (x).$$
Stable Currents in Projective Space.

Let $\mathcal{H}$ denote the space of holomorphic vector fields on $\mathbb{P}^n(\mathbb{C})$.
Let $S$ be an integral current on $\mathbb{P}^n(\mathbb{C})$.
Define a quadratic form $Q_S$ on $\mathcal{H}$ by

$$Q_S(V) = \left. \frac{d^2}{dt^2} M \{(\varphi_t)_* S\} \right|_{t=0}.$$ 

**Theorem. (Lawson-Simons)**

$$\text{trace} (Q_S) = - \int \| J(\vec{S}_x) \|^2 \|S\| (x)$$

**Corollary (Using Harvey-Shiffman).**

Every stable integral current in $\mathbb{P}^n(\mathbb{C})$ is an algebraic cycle.